Deformations of holographic duals to non-relativistic CFTs

JHEP07(2009)098

(http://iopscience.iop.org/1126-6708/2009/07/098)

Published by IOP Publishing for SISSA

Received: *May 30, 2009*
Accepted: *July 13, 2009*
Published: *July 30, 2009*

Deformations of holographic duals to non-relativistic CFTs

Nikolay Bobev[a,b] and Arnab Kundu[b]

[a] *Kavli Institute for Theoretical Physics, University of California Santa Barbara, Santa Barbara, CA 93106-4030, U.S.A.*

[b] *Department of Physics and Astronomy, University of Southern California, Los Angeles, CA 90089, U.S.A.*

E-mail: bobev@usc.edu, akundu@usc.edu

Abstract: We construct the non-relativistic counterparts of some well-known supergravity solutions dual to relevant and marginal deformations of $\mathcal{N} = 4$ super Yang-Mills. The main tool we use is the null Melvin twist and we apply it to the $\mathcal{N} = 1$ and $\mathcal{N} = 2^*$ Pilch-Warner RG flow solutions as well as the Lunin-Maldacena solution dual to β-deformations of $\mathcal{N} = 4$ super Yang-Mills. We also obtain a family of supergravity solutions with Schrödinger symmetry interpolating between the non-relativistic version of the $\mathcal{N} = 1$ Pilch-Warner and Klebanov-Witten fixed points. A generic feature of these non-relativistic backgrounds is the presence of non-vanishing internal fluxes. We also find the most general, three-parameter, null Melvin twist of the $\mathrm{AdS}_5 \times S^5$ black hole. We briefly comment on the field theories dual to these supergravity solutions.

Keywords: Gauge-gravity correspondence, AdS-CFT Correspondence

ArXiv ePrint: 0904.2873

doi:10.1088/1126-6708/2009/07/098

JHEP07(2009)098

Contents

1 Introduction and motivation

The gauge/gravity duality is a powerful tool for understanding strongly coupled field theories and it has been applied extensively to a plethora of relativistic conformal field theories in various dimensions [1, 2]. In condensed matter physics there exist strongly coupled systems which respect the non-relativistic analog of the conformal group, known as the

JHEP07(2009)098

Schrödinger group. Recently it has been shown that one can realize certain gravity solutions which are invariant under the Schrödinger group and thus extend the gauge/gravity duality to non-relativistic systems. This provides a new opportunity to apply holographic techniques to condensed matter systems.[1]

The celebrated example of gauge/gravity duality (in its weak form) is the equivalence between classical type IIB supergravity in $\mathrm{AdS}_5 \times S^5$ background and the large 't Hooft coupling limit of the planar $\mathcal{N} = 4$ super Yang-Mills (SYM) theory. It has been further realized that this correspondence can be extended to a large class of deformations of $\mathcal{N} = 4$ SYM which possess less global symmetry and supersymmetry. These include relevant deformations [28, 29] triggered by turning on certain mass terms for the adjoint scalars[2] and the marginal β-deformations [30, 31] corresponding to deformation of the $\mathcal{N} = 4$ superpotential. It seems natural to look for non-relativistic cousins of these gravity solutions and this will be our goal in this note.

Gravity solutions invariant under the Schrödinger group were first constructed in five dimensions [3, 4] and later embedded in ten dimensions by performing a null Melvin twist on the familiar $\mathrm{AdS}_5 \times S^5$ solution [8–10]. The null Melvin twist [32, 33] is a solution generating technique which can be applied to any solution of ten-dimensional supergravity with one compact and one non-compact U(1) isometries. It consists of performing a boost, T-duality, shift, T-duality and another boost. If the compact manifold has at least $\mathrm{U}(1)^3$ isometry (which is the case of S^5) one can generalize this transformation by allowing three independent shifts along the three independent U(1) directions. Using this we obtain the most general null Melvin twist of the $\mathrm{AdS}_5 \times S^5$ black hole solution. When all three shifts are set equal the background reduces to the one found in [8–10] and at zero temperature the background is invariant under the full Schrödinger group.

We also exploit the null Melvin twist to generate holographic duals to the non-relativistic version of $\mathcal{N} = 4$ SYM deformed by relevant and marginal operators. There are two RG flows that have explicit ten-dimensional gravity duals. The solution of [34] corresponds to an $\mathcal{N} = 1$ supersymmetric RG flow triggered by a mass term for one of the three complex chiral superfields of $\mathcal{N} = 4$ SYM. This solution flows to an $\mathcal{N} = 1$ superconformal fixed point in the IR and the gravity dual of this conformal fixed point was originally found in [35]. The other type IIB background that we consider is dual to the $\mathcal{N} = 2^*$ supersymmetric RG flow induced by turning on masses for two of the chiral superfields of $\mathcal{N} = 4$ SYM. The gravity dual of this flow was found in [36] (see also [37]). In the infrared it does not flow to a fixed point, instead there is a particular distribution of D3 branes which is interpreted as the Coulomb branch of the gauge theory.[3] We also find the non-relativistic generalizations of the one parameter family of gravity solutions interpolating between the Klebanov-Witten(KW) point [38] and a $\mathbb{Z}_2$ orbifold of the Pilch-Warner(PW) [35] fixed point [39]. Finally we consider the non-relativistic cousin of the Lunin-Maldacena solution [31] dual to the β-deformation of $\mathcal{N} = 4$ SYM. All these solutions

[1] See [3]–[26] for a sample of recent work and [27] for a review and a more complete list of references.

[2] See [2] for a review on holographic RG flows and a more complete list of references.

[3] We will make a slight abuse of language here and use the term non-relativistic Coulomb branch flow to describe the non-relativistic version of the Coulomb branch RG flow of the SYM.

have at least one U(1) isometry on the compact manifold and we find their non-relativistic versions by applying the null Melvin twist along this U(1). An important general feature of the solutions of [31, 34, 36, 39] is the presence of NS and RR fluxes on the internal manifold. The null Melvin twist preserves these fluxes and to the best of our knowledge the solutions in section 4 and section 7 are the first examples of gravity backgrounds with Schrödinger isometry which have non-trivial internal flux.

It is worth mentioning that an RG flow geometry which flows from a relativistic fixed point to a non-relativistic Lifshitz-like fixed point has been constructed in [26]. In contrast, the RG flow background discussed in section 3 interpolates between two vacua invariant under the Schrödinger symmetry.

It should be noted that there is no known finite temperature version of the ten-dimensional $\mathcal{N} = 1$ and $\mathcal{N} = 2^*$ PW solutions, so we refrain from addressing finite temperature aspects of the geometries dual to the non-relativistic RG-flows.[4] The finite temperature Lunin-Maldacena solution is well known and we find its non-relativistic version by applying the null Melvin twist. It will be interesting to analyze the thermodynamics of this background and understand if the marginal deformation affects certain transport coefficients in the dual field theory.

We start in section 2 with a review of the Schrödinger symmetry and its dual realization in type IIB supergravity, we also construct the most general null Melvin twist of $\mathrm{AdS}_5 \times S^5$. We introduce the PW background in section 3 and in section 4 show that the Melvin twisted PW background possesses the Schrödinger symmetry in the UV and the IR. Section 5 contains the non-relativistic version of the family of fixed points interpolating between the $\mathbb{Z}_2$ orbifold of the PW fixed point and the KW fixed point. In section 6 we discuss the non-relativistic version of the $\mathcal{N} = 2^*$ Coulomb branch RG flow solution. We apply the null Melvin twist to the Lunin-Maldacena background in section 7 and in section 8 we present some comments about the field theories dual to the geometries that we construct via the null Melvin twist. Section 9 contains some concluding remarks and open problems. Various technical details are presented in the appendices.

2 Schrödinger symmetry

2.1 The algebra

The symmetry group of the Schrödinger equation in flat space is known as the Schrödinger symmetry. The corresponding algebra is generated by spatial translations P^i, temporal translation H, spatial rotations M^{ij}, Galilean boost K^i, the dilatation operator D, a special conformal transformation C and the Galilean mass M. The explicit form of the algebra is given by the following commutation relations ($i = 1, \ldots, d$, where d is the number

[4]There was some work on the finite temperature $\mathcal{N}^* = 2$ PW solution in [40], however the explicit ten dimensional solution is not known.

of spatial dimensions)

$$\begin{aligned}
\left[M^{ij},M^{kl}\right] &= i\left(\delta^{ik}M^{jl}+\delta^{jl}M^{ik}-\delta^{il}M^{jk}-\delta^{jk}M^{il}\right), \quad [M^{ij},P^k]=i\left(\delta^{ik}P^j-\delta^{jk}P^i\right),\\
\left[M^{ij},K^k\right] &= i\left(\delta^{ik}K^j-\delta^{jk}K^i\right), \qquad [D,P^i]=-iP^i\,, \qquad [D,K^i]=iK^i\,, \quad [D,H]=-2iH\\
\left[P^i,K^j\right] &= -i\delta^{ij}M\,, \qquad\qquad [D,C]=2iC\,, \qquad [H,C]=iD\,.
\end{aligned} \tag{2.1}$$

The last two commutation relations involving the special conformal transformation generator C can only be included when the dynamical exponent, characterizing the different scaling of space and time, is 2. In this case, also the commutator $[D,M]=0$. Therefore the states are simultaneous eigenstates of the dilatation and the mass operator.

The Schrödinger algebra in d-spatial dimensions can be obtained by the light cone reduction of the relativistic conformal algebra in $(d+1)$-spatial dimensions. This can be intuitively understood by noticing that the light cone reduction of the massless Klein-Gordon equation (which is conformal) gives the Schrödinger equation in free space, we refer to [3, 4] for further details. Here we will be interested in the case $d=2$, i.e. non-relativistic field theories in $2+1$ dimensions.

2.2 The dual geometry

In [3, 4], a corresponding five-dimensional gravitational background was constructed which possesses the Schrödinger isometry group in two spatial dimensions. It was further realized in [8–10] that it is possible to embed this geometry in ten dimensions. This can be done by applying a solution generating technique, the null Melvin twist, to the well-known $\mathrm{AdS}_5\times S^5$ background (in the Poincare patch) [32, 33]. Here we will apply the most general null Melvin twist and generalize the background found in [8–10].

We start with the planar $\mathrm{AdS}_5\times S^5$ non-extremal black hole solution in string frame (The radii of the AdS_5 and S^5 are equal to L)

$$ds^2 = L^2r^2[-F(r)dt^2+dy^2+dx_1^2+dx_2^2]+\frac{L^2}{r^2F(r)}dr^2+L^2\sum_{i=1}^{3}(d\mu_i^2+\mu_i^2d\varphi_i^2)\,, \tag{2.2}$$

$$F_{(5)} = L^4(r^3dt\wedge dx_1\wedge dx_2\wedge dy\wedge dr+\sin^3\vartheta\cos\vartheta\sin\xi\cos\xi d\vartheta\wedge d\xi\wedge d\varphi_1\wedge d\varphi_2\wedge d\varphi_3)\,, \tag{2.3}$$

where

$$\mu_1=\cos\vartheta\,, \qquad \mu_2=\sin\vartheta\cos\xi\,, \qquad \mu_3=\sin\vartheta\sin\xi\,,$$

and

$$F(r)=1-\frac{r_+^4}{r^4}\,.$$

The coordinates on S^5 are chosen such that the U(1) isometries along ϕ_i are the $\mathrm{U}(1)^3$ Cartan subgroup of SO(6). Now we can apply the null Melvin twist to this background. The procedure is straightforward to implement and amounts to the following operations: first we boost in the (t,y) plane with parameter γ_0, then we perform a T-duality along y, then we shift all three Cartan angles of S^5 by $\varphi_i\to\varphi_i+a_iy$, then we perform another

T-duality along y and finally we perform an inverse boost in the (t, y) plane with parameter $-\gamma_0$ and take the limit $a_i \to 0$, $\gamma_0 \to \infty$ such that $a_i \cosh\gamma_0 = a_i \sinh\gamma_0 =$ finite. We also define $\eta_i \equiv a_i \cosh\gamma_0 = a_i \sinh\gamma_0$. Note that we are doing something a bit more general than the transformation in [8–10] where the case $a_1 = a_2 = a_3$ was considered which corresponds to null Melvin twist along the Hopf fiber of S^5. One can think of the third step of the null Melvin twist as three simultaneous shifts in all three Cartan directions (or a TsssT transformation for short [31, 41]). This general null Melvin twist generates a metric of the form

$$\begin{aligned} ds^2 =& L^2 r^2 \left[-\frac{r^2 F(r) q(\vartheta,\xi)}{K(r)}(dt+dy)^2 - \frac{F(r)}{K(r)}dt^2 + \frac{dy^2}{K(r)} + dx_1^2 + dx_2^2 \right] + \frac{L^2}{r^2 F(r)} dr^2 \\ &+ L^2 \sum_{i=1}^{3} \left[d\mu_i^2 + \mu_i^2 d\varphi_i^2 \right] - \frac{L^2 r_+^4}{r^2 K(r)} \left(\sum_{i=1}^{3} L^2 \eta_i \mu_i^2 d\varphi_i \right)^2 , \end{aligned} \tag{2.4}$$

as well as an NS two-form and a non-trivial dilaton given by

$$B_{(2)} = \frac{L^2 r^2}{K(r)} \left(\sum_{i=1}^{3} \eta_i \mu_i^2 d\varphi_i \right) \wedge (F(r)dt + dy) , \qquad \Phi(r) = -\frac{1}{2} \log K(r) , \tag{2.5}$$

where we have defined[5]

$$K(r) = 1 + \frac{r_+^4}{r^2} q(\vartheta,\xi) , \qquad \text{and} \qquad q(\vartheta,\xi) = L^4 \sum_{i=1}^{3} \eta_i^2 \mu_i^2 .$$

The self-dual five-form flux $F_{(5)}$ remains unaffected. This is the most general null Melvin twist of the non-extremal D3 brane solution of which we can now take various limits.

First let us consider the zero temperature (extremal) limit which amounts to setting $r_+ = 0$. The background simplifies to

$$\begin{aligned} ds^2 =& -\frac{L^2 q(\vartheta,\xi)}{z^4} du^2 + \frac{L^2}{z^2} \left(-2du dv + dx_1^2 + dx_2^2 + dz^2 \right) + L^2 \sum_{i=1}^{3} (d\mu_i^2 + \mu_i^2 d\varphi_i^2) , \\ B_{(2)} =& \frac{L^2}{z^2} \left(\sum_{i=1}^{3} \eta_i \mu_i^2 d\varphi_i \right) \wedge du , \end{aligned} \tag{2.6}$$

where we have defined new coordinates

$$u = t + y , \qquad v = \frac{1}{2}(t - y) , \qquad z = \frac{1}{r} . \tag{2.7}$$

The dynamical exponent of the solution, ν, parametrizes the different scaling of time and space in the dual non-relativistic theory and is determined by the g_{uu} term in the metric, $g_{uu} \sim z^{-2\nu}$. Our solutions have $\nu = 2$, this is expected since only non-relativistic systems with $\nu = 2$ admit the full Schrödinger symmetry. Indeed, following [3], one can verify that the background above has the full Schrödinger symmetry. The generators of the

[5]For generic values of η_i the function K depends on ϑ and ξ, for brevity we will denote it just by $K(r)$.

JHEP07(2009)098

Schrödinger algebra are given by the following isometries of the metric (ϵ and ϵ_i $(i = 1, 2)$ are infinitesimal parameters)

$$
\begin{aligned}
P^i &: x_i \to x_i + \epsilon_i \,, \quad H : u \to u + \epsilon \,, \qquad M : v \to v + \epsilon \,, \\
K^i &: x_i \to x_i - \epsilon_i u \,, \quad v \to v - \epsilon_i x_i \,, \quad M_{12} : x_1 \to x_1 + \epsilon x_2 \,, \quad x_2 \to x_2 - \epsilon x_1 \,, \\
D &: x_i \to (1-\epsilon) x_i \,, \quad z \to (1-\epsilon) z \,, \qquad u \to (1-\epsilon)^2 u \,, \quad v \to v \,, \\
C &: x_i \to (1-\epsilon u) x_i \,, \quad z \to (1-\epsilon u) z \,, \qquad u \to (1-\epsilon u) u \,, \quad v \to v - \frac{\epsilon}{2}(x_i x_i + z^2) \,.
\end{aligned}
\tag{2.8}
$$

One can easily check that the zero temperature background above is invariant under these infinitesimal transformations. Alternatively one can show that the five-dimensional non-compact metric has nine Killing vectors and their Lie brackets close under the Schrödinger algebra. It should be possible to construct generalizations of the solution (2.6) with different dynamical exponents, $\nu > 2$, along the lines of [15], the field theories dual to such solutions will not be symmetric under special conformal transformations since these are present only for $\nu = 2$ [42].

It is interesting to consider also the case $\eta = \eta_1 = \eta_2 = \eta_3$, then the extremal solution simplifies even further to

$$
\begin{aligned}
ds^2 &= -\frac{L^6 \eta^2}{z^4} du^2 + \frac{L^2}{z^2}\left(-2du dv + dx_1^2 + dx_2^2 + dz^2\right) \\
&\quad + L^2 \left(d\mu^2 + \frac{\sin^2\mu}{4}(\sigma_1^2 + \sigma_2^2) + \frac{\sin^2\mu \cos^2\mu}{4}\sigma_3^2 + \left(d\psi + \frac{\sin^2\mu}{2}\sigma_3 \right)^2 \right) ,
\end{aligned}
\tag{2.9}
$$

$$
B_{(2)} = \frac{L^2 \eta}{z^2}\left(d\psi + \frac{\sin^2\mu}{2}\sigma_3 \right) \wedge du \ , \tag{2.10}
$$

where the metric on S^5 has been written as a Hopf fiber[6] over $\mathbb{CP}^2$, σ_i are the left invariant SU(2) one forms

$$
\begin{aligned}
\sigma_1 &= \cos\alpha_3 \ d\alpha_1 + \sin\alpha_1 \ \sin\alpha_3 \ d\alpha_2 \,, \\
\sigma_2 &= \sin\alpha_3 \ d\alpha_1 - \sin\alpha_1 \ \cos\alpha_3 \ d\alpha_2 \,, \\
\sigma_3 &= d\alpha_3 + \cos\alpha_1 \ d\alpha_2 \,,
\end{aligned}
\tag{2.11}
$$

and

$$
J = \frac{1}{2} d\mathcal{A} \equiv \frac{1}{4}\ d\left(\sin^2\mu \ \sigma_3\right)
$$

is the Kähler form on $\mathbb{CP}^2$. This is the type IIB background constructed in [8–10] and as we already emphasized it is a special case of the more general null Melvin twist of $\mathrm{AdS}_5 \times S^5$ given in (2.4), (2.5).

The Galilean mass in the Schrödinger algebra, M, can be thought of as the number density in the non-relativistic field theory and is identified with the momentum along the compact v direction, $P_v = \frac{M}{R_v}$, where R_v is the radius of the v circle [8–10]. The black hole solution after the twist has entropy, temperature and a chemical potential conjugate to P_v. In addition to that we have the three deformation parameters η_i representing the

[6] We refer to appendix D for the explicit coordinate change.

freedom to choose the twist parameters for the three possible U(1) isometries which are global symmetries in the dual field theory. The overall scale associated with these three twist parameters is related to the chemical potential. In the case in which all three η_i vanish we have a DLCQ of AdS_5 (or AdS_5-BH) corresponding to zero number density - i.e., no particles in the non-relativistic theory [10].

An important feature of the general null Melvin twist is the appearance of the function $q(\vartheta, \xi)$ in front of du^2. For generic values of η_i this function is positive definite and thus there are no singularities or causal pathologies [43, 44]. However for special choices of η_i, $q(\vartheta, \xi)$ may have zeroes. We have checked explicitly that the curvature of the ten dimensional solution is finite for all values of η_i. Functions which depend on the internal manifold do appear in the g_{uu} component of the metric in some of the solutions analyzed in [15], however in [15] these are eigenfunctions of the Laplacian on the internal manifold and thus necessarily change sign, which may lead to problems with stability and causality.[7]

The zero temperature background in equations (2.2), (2.3) before the general null Melvin twist has 32 Killing spinors (and thus 32 supercharges) and one can show that for generic values of η_i none of these Killing spinors is preserved by the non-relativistic background given in equations (2.4), (2.5). However there are special values of η_i for which some supercharges are preserved [42].

3 The Pilch-Warner flow

We start with the Pilch-Warner flow solution presented in [34, 46]. We use the metric in [46] since it is written in a more convenient way. Note that for convenience we make a shift of σ_3 as compared to [46], namely $\sigma_3^{\text{here}} \to \sigma_3^{\text{there}} + d\phi$. This makes the coordinate ϕ the U(1) R-symmetry direction and the vielbein is

$$\begin{aligned}
e^{\mu+1} &= \Omega e^A dx^\mu\,, \qquad \mu = 0,1,2,3\,,\\
e^5 &= \Omega dr\,,\\
e^6 &= L\frac{\Omega\rho^2}{X_1}\left[\left(1-\frac{3}{2}\cos^2\theta\right)d\phi + \frac{1}{2}\cos^2\theta\sigma^3\right]\,,\\
e^7 &= L\frac{\Omega}{\rho\cosh\chi}d\theta\,,\\
e^8 &= L\frac{\Omega}{\rho\cosh\chi}\sin\theta\cos\theta\left[\left(\frac{3}{2}+\frac{1-\rho^6}{X_1}\left(1-\frac{3}{2}\cos^2\theta\right)\right)d\phi - \frac{1}{2}\left(1-\frac{1-\rho^6}{X_1}\cos^2\theta\right)\sigma_3\right],\\
e^9 &= L\frac{\rho}{2\Omega}\cos\theta\;\sigma_1\,,\\
e^{10} &= L\frac{\rho}{2\Omega}\cos\theta\;\sigma_2\,,
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
X_1 &= \cos^2\theta + \rho^6\sin^2\theta\,,\\
\Omega &= \frac{(\cosh\chi)^{1/2}X_1^{1/4}}{\rho^{1/2}}\,.
\end{aligned} \tag{3.2}$$

[7]We thank Mukund Rangamani for useful comments on this point.

The functions $\rho(r)$ and $\chi(r)$ are the two supergravity scalars that trigger the flow and σ_i are the SU(2) left-invariant one forms explicitly written in (2.11). There is a non-zero complex two-form potential

$$\mathcal{B} = \frac{i}{2} \sinh \chi (e^7 - ie^8) \wedge (e^9 - ie^{10}) \,,$$

where the NS and RR 2-forms are given by

$$B_{(2)} = \mathrm{Re}(\mathcal{B}) \,, \qquad\qquad C_{(2)} = \mathrm{Im}(\mathcal{B}) \;. \tag{3.3}$$

The self-dual five-form flux is given by

$$F_{(5)} = (1+\star) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge (w_r(r,\theta)dr + w_\theta(r,\theta)d\theta) \,, \tag{3.4}$$

where

$$\begin{aligned} w_r(r,\theta) &= \frac{e^{4A}}{4L} \frac{\cosh^2\chi}{\rho^4} \left[(\cosh(2\chi)-3)\cos^2\theta + \rho^6(2\rho^6 \sinh^2\chi \sin^2\theta + \cos(2\theta) - 3)\right] , \\ w_\theta(r,\theta) &= \frac{e^{4A}}{8\rho^2} \left[2\cosh^2\chi + \rho^6(\cosh(2\chi)-3)\right] \sin(2\theta) \,. \end{aligned} \tag{3.5}$$

The scalar flow has two critical points:

1) $\mathrm{AdS}_5 \times \mathrm{S}^5$, this is the UV critical point given by $\chi = 0$, $\rho = 1$ and $A = \dfrac{r}{L} \equiv \dfrac{r}{L_{UV}}$.

2) $\mathrm{AdS}_5 \times X_{PW}$, this is the IR critical point (Pilch-Warner fixed point) and is given by $\chi = \dfrac{\log(3)}{2}$, $\rho = 2^{1/6}$ and $A = \dfrac{2^{5/3}}{3}\dfrac{r}{L} \equiv \dfrac{r}{L_{IR}}$.

Note that as required by the holographic c-theorem [28] the radius of AdS_5 decreases along the flow

$$\frac{L_{IR}}{L_{UV}} = \frac{3}{2^{5/3}} \approx 0.944941 \;.$$

The metric of this RG flow solution is $\mathrm{SU}(2) \times \mathrm{U}(1)_\phi \times \mathrm{U}(1)_{\alpha_3}$ invariant, however the complex two form breaks this down to $\mathrm{SU}(2) \times \mathrm{U}(1)_\phi$ because

$$\mathcal{B} \sim (\sigma_1 - i\sigma_2) \sim e^{-i\alpha_3} \;. \tag{3.6}$$

Since the background along the entire flow is manifestly $\mathrm{SU}(2) \times \mathrm{U}(1)_\phi$ invariant one can easily apply the null Melvin twist along the two U(1) isometries $y \equiv x_3$ and ϕ. This is done explicitly in appendix A.

It is worth noting that $\mathrm{U}(1)_\phi$ is not the Hopf fiber and one can show that at the UV fixed point the particular null Melvin twist that we apply to the PW solution corresponds to the following choice of the twist parameters

$$\eta_1 = \eta \,, \qquad \eta_2 = -\frac{\eta}{2} = \eta_3 \;. \tag{3.7}$$

The full RG flow solution after the null Melvin twist is not particularly illuminating so we present its explicit form in appendix A and we proceed with a discussion of the non-relativistic version of the flow geometry at the fixed points.

4 Fixed points and Schrödinger symmetry

The supergravity scalar flow after the twist still has two fixed points and as we show in this section the supergravity backgrounds at these fixed points posses the full Schrödinger symmetry.

4.1 UV fixed point

As mentioned in the previous section, at the UV fixed point we have

$$\rho = 1\,, \qquad \chi = 0\,, \qquad \Omega = 1\,, \qquad X_1 = 1\,, \qquad A = \frac{r}{L}\,. \tag{4.1}$$

The metric, after the null Melvin twist, takes the form

$$\begin{aligned} ds_{10}^2 &= -L^2 e^{4r/L}\left(1 - \frac{3}{4}\cos^2\theta\right)[\eta(dt+dy)]^2 - e^{2r/L}(dt-dy)(dt+dy) \\ &\quad + e^{2r/L}(dx_1^2 + dx_2^2) + dr^2 + L^2 ds_{UV}^2\,, \end{aligned}$$

where ds_{UV}^2 is the metric on the S^5 and is given by

$$ds_{UV}^2 = d\theta^2 + \frac{\cos^2\theta}{4}(\sigma_1^2+\sigma_2^2) + \frac{\sin^2\theta\cos^2\theta}{4}(3d\phi - \sigma_3)^2 + \left(\left(1 - \frac{3}{2}\cos^2\theta\right)d\phi + \frac{\cos^2\theta}{2}\sigma_3\right)^2. \tag{4.2}$$

The NS two-form is

$$B_{(2)} = L^2 e^{2r/L}\left(\left(1 - \frac{3}{4}\cos^2\theta\right)d\phi - \frac{\cos^2\theta}{4}\,\sigma_3\right)\wedge[\eta(dt+dy)]\,. \tag{4.3}$$

Now define new coordinates

$$u = (t+y)\,, \qquad v = \frac{1}{2L^2}(t-y)\,, \qquad z = e^{-r/L}\,, \qquad \hat{x}_1 = L^{-1}x_1\,, \qquad \hat{x}_2 = L^{-1}x_2\,.$$

In this coordinate the metric and the B-field becomes

$$\begin{aligned} ds_{10}^2 &= L^2\left(-\left(1 - \frac{3}{4}\cos^2\theta\right)\frac{\eta^2}{z^4}du^2 + \frac{1}{z^2}\left(-2dudv + d\hat{x}_1^2 + d\hat{x}_2^2 + dz^2\right)\right) + L^2 ds_{UV}^2. \\ B_{(2)} &= \frac{\eta L^2}{z^2}\left(\left(1 - \frac{3}{4}\cos^2\theta\right)d\phi - \frac{\cos^2\theta}{4}\,\sigma_3\right)\wedge du. \end{aligned} \tag{4.4}$$

The RR sector in the UV is completely invariant under the null Melvin twist.

Note the appearance of a function of θ in front of the du^2 term in the metric. As noted before, this can be traced back to the fact that we did not use the Hopf fiber of S^5 for the null Melvin twist. It is obvious that the function in front of the du^2 term in the metric above is negative definite. Therefore there are no spacetime pathologies related to causality and instability of this background [43, 44]. As expected the dynamical exponent of this solution is 2.

4.2 IR fixed point

Now we move to the analysis of the IR fixed point. This fixed point is given by

$$\rho = 2^{1/6}\,, \qquad \chi = \frac{\log 3}{2}\,, \qquad \Omega = \frac{2^{5/12}}{3^{1/4}}(1+\sin^2\theta)^{1/4}\,, \qquad X_1 = 1+\sin^2\theta\,, \qquad A = \frac{r}{L_{IR}}\,,$$

where

$$L_{IR} = \frac{3}{2^{5/3}}L\,. \tag{4.5}$$

The metric has the form

$$\begin{aligned} ds_{10}^2 = \Omega^2 L_{IR}^2 \Bigg[& -6\left(\frac{1+3\sin^4\theta}{1+\sin^2\theta}\right)\frac{\eta^2}{z^4}du^2 \;+\; \frac{1}{z^2}\left(-2dudv \;+\; d\hat{x}_1^2 + d\hat{x}_2^2 \;+\; dz^2\right) \\ & -2\eta\frac{\cos^2\theta\sin\theta}{1+\sin^2\theta}\,\sigma_1\frac{du}{z^2}\Bigg] + ds_{PW}^2\,, \end{aligned} \tag{4.6}$$

where again we have defined new coordinates

$$u = \frac{2^{4/3}}{3^2}(t+y)\,, \quad v = \frac{3^2}{2^{4/3}}\frac{1}{2L_{IR}^2}(t-y)\,, \quad z = e^{-r/L_{IR}}\,, \quad \hat{x}_1 = L_{IR}^{-1}x_1\,, \quad \hat{x}_2 = L_{IR}^{-1}x_2\,,$$

and ds_{PW}^2 is the metric on the deformed S^5 at the Pilch-Warner fixed point[8] and is given by

$$\begin{aligned} ds_{PW}^2 = L_{IR}^2\frac{2^{\frac{5}{6}}}{3^{\frac{3}{2}}}(1+\sin^2\theta)^{\frac{1}{2}}\Bigg[& 2d\theta^2 + \frac{\cos^2\theta}{1+\sin^2\theta}(\sigma_1^2+\sigma_2^2) + 8\frac{\sin^2\theta\cos^2\theta}{(1+\sin^2\theta)^2}\left(d\phi - \frac{\sigma_3}{2}\right)^2 \\ & + \frac{16}{3}\frac{1}{(1+\sin^2\theta)^2}\left(\left(1-\frac{3}{2}\cos^2\theta\right)d\phi + \frac{\cos^2\theta}{2}\sigma_3\right)^2\Bigg]. \end{aligned} \tag{4.7}$$

We again note that the function in front of the du^2 term in the metric is negative definite, hence the IR fixed point is also free of spacetime pathologies. The NS two-form is

$$B_{(2)} = B_1 + B_{PW}\,,$$

where B_1 is the piece of the B-field generated by the null Melvin twist

$$B_1 = \eta L_{IR}^2\frac{2^{\frac{7}{3}}}{3}\left(\frac{1+3\sin^4\theta}{1+\sin^2\theta}\,d\phi - \frac{\cos^2\theta}{1+\sin^2\theta}\,\sigma_3\right)\wedge\frac{du}{z^2}\,, \tag{4.8}$$

and B_{PW} is the usual internal B-field of the Pilch-Warner fixed point solution [35]

$$B_{PW} = \frac{2^{\frac{4}{3}}}{3^2}L_{IR}^2\left[\frac{\cos^2\theta\sin\theta}{1+\sin^2\theta}\left(d\phi - \frac{1}{2}\sigma_3\right)\wedge\sigma_1 + \frac{\cos\theta}{2}d\theta\wedge\sigma_2\right]. \tag{4.9}$$

The five-form RR flux is modified by the null Melvin twist and takes the form

$$\widetilde{F}_{(5)} = F_{(5)} + \eta L_{IR}^2\frac{2^{\frac{7}{3}}}{3}(1+\star)\frac{du}{z^2}\wedge\left(\frac{1+3\sin^4\theta}{1+\sin^2\theta}\,d\phi - \frac{\cos^2\theta}{1+\sin^2\theta}\,\sigma_3\right)\wedge dC_{(2)}\,. \tag{4.10}$$

The RR two-form $C_{(2)}$ remains unchanged.

[8]This fixed point solution was originally found (in slightly different coordinates) in [35], see also [45].

JHEP07(2009)098

It can be checked that background metric (4.6), the NS two-form (4.8) and the RR five-form (4.10) at the IR fixed point are invariant under the Schrödinger symmetry (2.8). However there is a new interesting feature. Recall that the null coordinate u is identified with the time coordinate for the non-relativistic field theory. Therefore the off-diagonal term in the metric between u and σ_1 can be interpreted as a rotation along the compact σ_1-direction. The presence of this term can be traced back to the fact that we had non-zero NS flux in the solution before the twist. It will be interesting to understand the meaning of this rotation term from the point of view of the dual field theory. It will be quite interesting to see if this Schrödinger invariant ten-dimensional solution can be reduced to five dimensions and understood as a solution to some effective five-dimensional equations of motion. It is not immediately clear to us that this is possible and we will think of the whole RG flow solution and the IR fixed point as purely ten-dimensional.

The metric of the relativistic Pilch-Warner flow geometry has an U(1) isometry which rotates σ_1 in to σ_2 (this is the $U(1)_{\alpha_3}$), however, this is not a symmetry of the background because of the non-zero two-form (3.6). The null Melvin twist enhances the breaking of this $U(1)_{\alpha_3}$ by generating an off-diagonal term in the metric. This enhanced symmetry breaking is present everywhere along the flow and at the IR fixed point but vanishes in the UV.

The dynamical exponent of this solution is $\nu = 2$, so we see that along the RG flow the dynamical exponent is invariant. It should be possible to find similar gravity solutions with different dynamical exponents along the lines of [15]. To do this one should make a more general ansatz with $g_{uu} \sim z^{-2\nu}$ and solve the equations of motion. Due to the presence of internal fluxes this will be a non-trivial task.

5 Family of non-relativistic fixed points

There is a one parameter family of supergravity fixed point solutions, [39], which interpolate between the $\mathbb{Z}_2$ orbifold of the PW fixed point [35] and the $AdS_5 \times T^{(1,1)}$ solution discussed by Klebanov-Witten in [38] and originally found by Romans [47]. These solutions are gravity duals to a family of $\mathcal{N} = 1$ conformal quiver gauge theories which can be thought of as IR fixed points of mass deformations of a $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM. An interesting feature of the interpolating family is that the axion-dilaton vanishes at both the PW and the KW fixed points but has a non-trivial dependence on the coordinates of the internal manifold for all interpolating solutions. Here we will apply the null Melvin twist to the family of solutions in [39] and generate a new family of non-relativistic fixed points invariant under the Schrödinger symmetry. We should note that the analytic form of this family of solutions is not known, however in [39] numerical solutions to the supersymmetry equations were found and it was shown explicitly that they interpolate between the KW and PW solutions. More details on the solutions of [39] are given in appendix B.[9]

The structure of the metric and the two-forms given in appendix B is almost identical to the ones for the PW flow solutions from section 4. The only difference is the $d\alpha_1 d\alpha_2$ term in the metric which was not present in the PW solution, however one can show that

[9]Note that we use“hats”,e.g. $\hat{f}_i, \hat{b}_i$, to indicate that these are different functions from the one used in appendix A.

this term does not modify the result of the null Melvin twist so we can easily find the non-relativistic analog of the family of fixed points found in [39]. Since we have gravity duals to a family of fixed points the function $A(r)$ is simply

$$A(r) = \frac{r}{L_{IR}}\,, \tag{5.1}$$

the constant f_0 defined in [39] can be written as

$$f_0 = \frac{2^{5/3}}{3}\,. \tag{5.2}$$

The null Melvin twist modifies the metric and the B-field and leaves the dilaton invariant

$$\begin{aligned} ds^2_{10} &= \hat{\Omega}^2 L^2_{IR}\left(-\frac{\eta^2 \hat{f}_4 \hat{\Omega}^2}{L^2_{IR}}\frac{du^2}{z^4} + \frac{1}{z^2}\left(-2dudv + d\hat{x}_1^2 + d\hat{x}_2^2 + dz^2\right)\right.\\ &\qquad\left. -2\frac{\hat{b}_1}{L^2_{IR}\cos\alpha_3}\,\sigma_1\frac{du}{z^2}\right) + ds^2_{\text{int}}\,,\\ B_{(2)} &= \eta\hat{\Omega}^2(\hat{f}_4 d\phi + \hat{f}_6\sigma_3)\wedge\frac{du}{z^2} + B_{(2)\text{int}}\,, \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} ds^2_{\text{int}} &= \hat{f}_8 d\theta^2 + \hat{f}_1\, d\alpha_1^2 + 2\hat{f}_9\, d\alpha_1 d\alpha_2 + \hat{f}_2\, d\alpha_2^2 + \hat{f}_3\, d\alpha_3^2\\ &\quad + \hat{f}_4\, d\phi^2 + 2\hat{f}_5\, d\alpha_2 d\alpha_3 + 2\hat{f}_6\, d\alpha_3 d\phi + 2\hat{f}_7\, d\alpha_2 d\phi\,, \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} B_{(2)\text{int}} &= \hat{b}_1\, d\phi\wedge d\alpha_1 + \hat{b}_2\, d\alpha_3\wedge d\alpha_1 + \hat{b}_3\, d\alpha_2\wedge d\alpha_1 + \hat{b}_4\, d\theta\wedge d\alpha_2\\ &\quad + \hat{b}_5\, d\theta\wedge d\alpha_1 + \hat{b}_6\, d\phi\wedge d\alpha_2 + \hat{b}_7\, d\alpha_3\wedge d\alpha_2\,. \end{aligned} \tag{5.5}$$

Again we have performed the change of variables

$$u = (t+y)\,,\qquad v = \frac{1}{2L^2_{IR}}(t-y)\,,\qquad z = e^{-r/L_{IR}}\,,\qquad \hat{x}_1 = L^{-1}_{IR}x_1\,,\qquad \hat{x}_2 = L^{-1}_{IR}x_2\,.$$

The RR two-form is invariant under the null Melvin twist and the self-dual five-form transforms to

$$\widetilde{F}_{(5)} = F_{(5)} + \eta\,(1+\star)\left(\hat{\Omega}^2 e^{2A} du\wedge\left(\hat{f}_4 d\phi + \hat{f}_6 d\alpha_3 + \hat{f}_7 d\alpha_2\right)\right)\,. \tag{5.6}$$

Note that the functions $\hat{f}_4$, $\hat{f}_6$, $\hat{\Omega}$ and $\dfrac{\hat{b}_1}{\cos\alpha_3}$ depend only on θ. Their analytic form is not explicitly known for the whole family of fixed point solutions (they are known at the KW and PW points). However one can find numerical solutions for them [39].

The construction above shows that there is a one parameters family of supergravity solutions with Schrödinger symmetry interpolating between the non-relativistic cousins of the KW and PW fixed point solutions. It should be noted also that the function in front of the du^2-term in the metric, denoted by $(-\hat{\Omega}^4\hat{f}_4)$, is negative definite ensuring the absence spacetime pathologies for all backgrounds in the interpolating family.

5.1 The non-relativistic Klebanov-Witten point

At the KW point the background simplifies significantly and we have

$$\hat{b}_i = \hat{c}_i = 0\,, \qquad \hat{\Omega}^2 = f_0^{1/2}\,, \tag{5.7}$$

one finds also that

$$\hat{f}_4 = \frac{4}{9}L_{IR}^2 f_0^{1/2}\,, \qquad \hat{f}_6 = -\frac{4}{9}L_{IR}^2 f_0^{1/2}\cos\theta\ . \tag{5.8}$$

The metric and the B-field are then

$$\begin{aligned} ds_{10}^2 &= \frac{2^{5/6}}{3^{1/2}}L_{IR}^2\left(-\frac{\eta^2}{z^4}du^2 + \frac{1}{z^2}\left(-2dudv + d\hat{x}_1^2 + d\hat{x}_2^2 + dz^2\right)\right) + ds_{\text{int}}^2\,,\\ B_{(2)} &= \eta\frac{2^{11/6}}{3^{3/2}}L_{IR}^2(d\phi - \cos\theta\sigma_3)\wedge\frac{du}{z^2}\,, \end{aligned} \tag{5.9}$$

where

$$ds_{\text{int}}^2 = \frac{2^{5/6}}{3^{3/2}}L_{IR}^2\left[d\theta^2 + \cos^2\theta\sigma_1^2 + \sigma_2^2 + \frac{4\sin^2\theta}{3+\cos^2\theta}d\phi^2 + \frac{3+\cos^2\theta}{3}\left(\sigma_3 - \frac{4\cos\theta}{3+\cos^2\theta}d\phi\right)^2\right] \tag{5.10}$$

As expected from the general calculation about the whole family of fixed points this solution is invariant under the Schrödinger symmetry. Note that there is no off-diagonal $\sigma_1 du$ term in the metric and the coefficient in front of du^2 is a constant. There is a further simplification in this solution since the internal NS and RR fluxes vanish. These features are due to the fact that the internal manifold is $T^{(1,1)}$, which is a Sasaki-Einstein manifold and the coordinate ϕ used for the null Melvin twist happens to be the U(1) symmetry corresponding to the Reeb vector [8–10]. It is also possible to find the finite temperature version of this solution by putting a Schwarzschild black hole in the AdS_5. Due to the complications arising from the presence of internal fluxes and the non-trivial dilaton the finite temperature solutions for the rest of the family of fixed points are not known.

6 Non-relativistic Coulomb branch RG flows

It is natural to consider the null Melvin twist of RG flow solutions which do not flow to a fixed point in the IR. One such example for which the type IIB solution is explicitly known is the RG flow solution dual to $\mathcal{N} = 2^*$ SYM found in [36]. This geometry is dual to a particular mass deformation of $\mathcal{N} = 4$ SYM and the field theory flows to the Coulomb branch. The background has internal NS and RR fluxes and is thus different from the non-relativistic Coulomb branch gravity solutions discussed in [15].

The background geometry, in the notation of [36], can be written as

$$\begin{aligned} ds^2 &= \Omega^2 ds_{1,4}^2 + ds_5^2\,, \quad ds_5^2 = \frac{a^2}{2}\frac{(cX_1X_2)^{1/4}}{\rho^3}\left[\frac{d\theta^2}{c} + \frac{\rho^6}{4}\cos^2\theta\left(\frac{\sigma_1^2}{cX_2} + \frac{\sigma_2^2+\sigma_3^2}{X_1}\right) + \frac{\sin^2\theta}{X_2}d\phi^2\right]\,,\\ c &= \cosh 2\chi\,, \qquad \Omega^2 = \frac{(cX_1X_2)^{1/4}}{\rho}\,, \qquad X_1 = \cos^2\theta + \rho^6\cosh 2\chi\sin^2\theta\,,\\ X_2 &= \cosh 2\chi\cos^2\theta + \rho^6\sin^2\theta\,, \end{aligned} \tag{6.1}$$

where σ_i's are defined in (2.11) and $a = \sqrt{2}L_{UV}$. The metric has $\mathrm{SU}(2) \times \mathrm{U}(1)_{23} \times \mathrm{U}(1)_\phi$ isometry, where $\mathrm{U}(1)_{23}$ refers to rotating σ_2 to σ_3. As usual the dilaton-axion field can be written as a complex scalar field $\lambda = C_{(0)} + ie^{-\Phi}$ with

$$\lambda = i\left(\frac{1-b}{1+b}\right) , \quad b = \left(\frac{c^{1/2}X_1^{1/2} - X_2^{1/2}}{c^{1/2}X_1^{1/2} + X_2^{1/2}}\right) e^{2i\phi} . \tag{6.2}$$

This clearly breaks the $\mathrm{U}(1)_\phi$ and we are left with $\mathrm{SU}(2) \times \mathrm{U}(1)_{23}$ isometry.

The background also has non-zero RR and NS forms $C_{(2)}$,$C_{(4)}$ and $B_{(2)}$ given by

$$\begin{aligned}
&\mathcal{B}_{(2)} = e^{i\phi}\left(\frac{a_1}{2}\, d\theta\wedge\sigma_1 + \frac{a_2}{4}\,\sigma_2\wedge\sigma_3 + \frac{a_3}{2}\,\sigma_1\wedge d\phi\right) , \\
&\quad a_1 = -i\frac{a^2}{2}\tanh(2\chi)\cos\theta\,, \quad a_2 = i\frac{a^2}{2}\frac{\rho^6\sinh(2\chi)}{X_1}\sin\theta\cos^2\theta\,, \quad a_3 = \frac{a^2}{2}\frac{\sinh(2\chi)}{X_2}\sin\theta\cos^2\theta\,, \\
&B_{(2)} = \mathrm{Re}[\mathcal{B}_{(2)}]\,, \qquad\qquad C_{(2)} = \mathrm{Im}[\mathcal{B}_{(2)}]\,, \\
&C_{(4)} = e^{4A}\frac{X_1}{\rho^2}dt\wedge dx_1\wedge dx_2\wedge dy\,.
\end{aligned} \tag{6.3}$$

It is clear that the two-form potentials also break the $\mathrm{U}(1)_\phi$-isometry.

The UV fixed point is obtained by setting $\chi = 0$ and $\rho = 1$, where $A(r) = r/L$. To perform the null Melvin twist we choose a U(1) subgroup of the SU(2) isometry along the coordinate α_2. The metric after the twist is given by

$$\begin{aligned}
ds^2 &= -h\Omega^4 e^{4A}\left[\eta(dt+dy)\right]^2 + \Omega^2 e^{2A}\left[-(dt+dy)(dt-dy) + dx_1^2 + dx_2^2\right] + \Omega^2 dr^2 \\
&\quad + 2\Omega^2 e^{2A}\eta(dt+dy)\Sigma_{(1)} + ds_5^2\,,
\end{aligned} \tag{6.4}$$

where

$$\begin{aligned}
h &= \frac{a^2}{8}\left(cX_1X_2\right)^{1/4}\rho^3\cos^2\theta\left[\frac{1}{cX_2}\sin^2\alpha_1\sin^2\alpha_3 + \frac{1}{X_1}\left(\sin^2\alpha_1\cos^2\alpha_3 + \cos^2\alpha_1\right)\right] , \\
\Sigma_{(1)} &= \frac{a^2}{4}\sinh(2\chi)\cos\theta\left[\sin\alpha_1\sin\alpha_3\left(\frac{\sin\phi}{\cosh(2\chi)}d\theta - \frac{\cos\phi\sin\theta\cos\theta}{X_2}d\phi\right)\right. \\
&\qquad\left. - \frac{\rho^6}{2X_1}\sin\theta\cos\theta\sin\phi\left(\sin\alpha_1\cos\alpha_3 d\alpha_3 + \sin\alpha_3\cos\alpha_1 d\alpha_1\right)\right] .
\end{aligned} \tag{6.5}$$

The NS-NS two-form becomes

$$\begin{aligned}
&\widetilde{B}_{(2)} = \mathrm{Re}\left[\mathcal{B}_{(2)}\right] + \Omega^2 e^{2A}\Gamma_{(1)}\wedge\eta\,(dt+dy)\,, \\
&\Gamma_{(1)} = \frac{a^2}{8}\left(cX_1X_2\right)^{1/4}\rho^3\cos^2\theta\left[\frac{\cos\alpha_1}{X_1}d\alpha_3 + \sin\alpha_1\cos\alpha_3\sin\alpha_3\left(\frac{1}{cX_2} - \frac{1}{X_1}\right)d\alpha_1\right] + h d\alpha_2\,.
\end{aligned} \tag{6.6}$$

And finally the RR forms are given by

$$\begin{aligned}
\widetilde{\lambda} &= \lambda\,, \\
\widetilde{C}_{(2)} &= \mathrm{Im}\left[\mathcal{B}_{(2)}\right] + C_{(0)}\Sigma_{(1)}\wedge\eta\,(dt+dy)\ , \\
\widetilde{C}_{(4)} &= C_{(4)} - \Omega^2 e^{2A}C_{(2)}\wedge\Gamma_{(1)}\wedge\eta\,(dt+dy)\ .
\end{aligned} \tag{6.7}$$

From the definition of h in (6.5), it is clear that h is a positive function[10] except at $\theta = \pi/2$ and hence the corresponding non-relativistic background is again free off spacetime pathologies. At $\theta = \pi/2$ we have $g_{uu} = 0$, one can check however that the curvature is finite at this point so nothing dramatic happens to the ten-dimensional metric.

At the UV, the twisted background takes a pleasingly simple form

$$\begin{aligned}
ds^2 &= L^2\left(-\left(\frac{1}{4}\cos^2\theta\right)\frac{\eta^2}{z^4}du^2 + \frac{1}{z^2}\left(-2dudv + d\hat{x}_1^2 + d\hat{x}_2^2 + dz^2\right)\right) + L^2 ds^2_{UV}\,,\\
ds^2_{UV} &= d\theta^2 + \frac{1}{4}\cos^2\theta\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right) + \sin^2\theta d\phi^2\,,\\
\widetilde{B}_{(2)} &= \frac{\eta L^2}{z^2}\left(\frac{1}{4}\cos^2\theta\right)(d\alpha_2 + \cos\alpha_1 d\alpha_3)\wedge du\,,\\
\widetilde{C}_{(4)} &= \frac{L^2}{z^4}dt\wedge d\hat{x}_1\wedge d\hat{x}_2\wedge dy\,.
\end{aligned} \tag{6.8}$$

This background is invariant under the Schrödinger symmetry and is a special case of the general null Melvin twist of $\mathrm{AdS}_5\times S^5$ discussed in section 2 with

$$\eta_1 = 0\,, \qquad \eta_2 = -\eta_3 \equiv \frac{\eta}{2}\,. \tag{6.9}$$

As mentioned above the g_{uu} component of the metric has a zero at $\theta = \pi/2$, however the curvature of the ten dimensional solution is regular at this point so the background does not have physical singularities. We want to stress again that the main difference between the Coulomb branch solutions presented in [15] and the solution constructed above is the presence of the off-diagonal terms in the metric and the non-trivial $C_{(2)}$ and $B_{(2)}$ fluxes along the flow. The Coulomb branch solutions of [15] are non-relativistic cousins of the RG flows in [29], whereas what we have here is the non-relativistic version of the solution in [36].

Although the $\mathcal{N} = 2^*$ RG flow does not lead to a fixed point, the parent relativistic flow has a rich structure in the moduli space leading to the presence of the enhançon locus discussed in [48, 49]. In the relativistic case, this is obtained by probing the background geometry with a probe $D3$-brane which spreads out into the enhançon locus. Similar computation is directly amenable for the Melvin twisted non-relativistic background, however its dual field theory interpretation is unclear to us at present.

7 Non-relativistic Lunin-Maldacena solution

The last background to which we will apply the null Melvin twist is the Lunin-Maldacena deformation of $\mathrm{AdS}_5\times S^5$ [31]. The background is obtained from the extremal (or non-extremal [50]) D3 brane solution by applying S-duality, a T-duality a shift and another T-duality (TsT transformation) on two of the U(1) isometries of S^5 and then one more S-duality. The resulting solution (at zero temperature) is dual to the exactly marginal deformation of $\mathcal{N} = 4$ SYM discussed in [30]. There are three U(1) isometries on the S^5

[10] The functions $\cosh(2\chi)$ and $\rho = e^{\alpha(r)}$ are always positive for real $\chi(r)$ and real $\alpha(r)$. The family of different solutions for $\{\chi(r), \alpha(r)\}$ represents different flows to the $\mathcal{N} = 2^*$ gauge theory in the parent relativistic version. We refer to [36] for more details.

which survive the TsT transformation so in principle we can generate a three parameter non-relativistic background by applying the general null Melvin twist discussed in section 2. We will however restrict to the case of null Melvin twist along the Hopf fiber direction ψ which corresponds to the choice $\eta_1 = \eta_2 = \eta_3 \equiv \eta$. The complete untwisted Lunin-Maldacena solution at finite temperature is presented in appendix C, we refrain from presenting the detailed steps of the null Melvin twist and give just the final solution. We would like to point out that we consider the Lunin-Maldacena solution with complex deformation parameter $\beta = \gamma - i\sigma$, the special case of real deformation (known also as γ-deformation) can be obtained by setting $\sigma = 0$. In the case of real deformation one does not have to apply the S-duality and the background is obtained by a TsT transformation of $AdS_5 \times S^5$. We denote the metric and all the fields in the solution after the twist with a tilde

$$\begin{aligned}
\widetilde{ds}^2_\beta = \mathcal{H}^{1/2}L^2r^2\Big[&-\frac{\eta^2\mathcal{H}^{1/2}L^2r^2F}{1+\kappa f_4'}(dt+dy)^2 - \frac{F}{1+\kappa f_4'}dt^2 + \frac{1}{1+\kappa f_4'}dy^2 + dx_1^2 + dx_2^2 \\
&+\frac{dr^2}{Fr^4} - \frac{2\eta}{1+\kappa f_4'}(Fdt+dy)(b_0'd\mu+b_1'd\alpha_1+b_2'd\alpha_2+b_3'd\alpha_3)\Big] + \left(f_0'+\frac{\kappa b_0'^2}{1+\kappa f_4'}\right)d\mu^2 \\
&+\left(f_1'+\frac{\kappa b_1'^2}{1+\kappa f_4'}\right)d\alpha_1^2 + \left(f_2'+\frac{\kappa(b_2'^2-f_7'^2)}{1+\kappa f_4'}\right)d\alpha_2^2 + \left(f_3'+\frac{\kappa(b_3'^2-f_6'^2)}{1+\kappa f_4'}\right)d\alpha_3^2 \\
&+\frac{f_4'}{1+\kappa f_4'}\,d\psi^2 + \frac{2f_7'}{1+\kappa f_4'}\,d\psi d\alpha_2 + \frac{2f_6'}{1+\kappa f_4'}\,d\psi d\alpha_3 + 2\left(f_5' + \frac{\kappa(b_2'b_3'-f_6'f_7')}{1+\kappa f_4'}\right)d\alpha_2 d\alpha_3 \\
&+\frac{2\kappa}{1+\kappa f_4'}\left[b_0'b_1'd\mu d\alpha_1 + b_0'b_2'd\mu d\alpha_2 + b_0'b_3'd\mu d\alpha_3 + b_1'b_2'd\alpha_1 d\alpha_2 + b_1'b_3'd\alpha_1 d\alpha_3\right].
\end{aligned} \tag{7.1}$$

The B-field has two pieces - B_c which is generated by the Lunin-Maldacena transformation and has legs only along the compact manifold [31] and B_nc which is generated by the null Melvin twist

$$\widetilde{B}_{(2)} = B_\mathrm{c} + B_\mathrm{nc}\,, \tag{7.2}$$

where

$$\begin{aligned}
B_\mathrm{nc} &= -\frac{\mathcal{H}^{1/2}L^2\eta r^2}{1+\kappa f_4'}(Fdt+dy)\wedge(f_4'd\psi + f_6'd\alpha_3 + f_7'd\alpha_2)\,, \\
B_\mathrm{c} &= \frac{b_0'}{1+\kappa f_4'}d\psi\wedge d\mu + \frac{b_1'}{1+\kappa f_4'}d\psi\wedge d\alpha_1 + \frac{b_2'}{1+\kappa f_4'}d\psi\wedge d\alpha_2 + \frac{b_3'}{1+\kappa f_4'}d\psi\wedge d\alpha_3 \\
&\quad+\left(b_4' + \frac{\kappa b_0' f_6'}{1+\kappa f_4'}\right)d\mu\wedge d\alpha_3 + \left(b_5' + \frac{\kappa b_1' f_6'}{1+\kappa f_4'}\right)d\alpha_1\wedge d\alpha_3 \\
&\quad+\left(b_6' + \frac{\kappa(b_2'f_6' - b_3'f_7')}{1+\kappa f_4'}\right)d\alpha_2\wedge d\alpha_3 + \frac{\kappa b_0' f_7'}{1+\kappa f_4'}d\mu\wedge d\alpha_2 + \frac{\kappa b_1' f_7'}{1+\kappa f_4'}d\alpha_1\wedge d\alpha_2\,.
\end{aligned} \tag{7.3}$$

The dilaton is

$$\widetilde{\Phi} = \Phi - \frac{1}{2}\ln(1+\kappa f_4') = \frac{1}{2}\ln\left(\frac{\mathcal{G}\mathcal{H}^2}{1+\kappa f_4'}\right)\,. \tag{7.4}$$

JHEP07(2009)098

After the null Melvin twist the RR potentials become

$$\widetilde{C}_{(0)} = C_{(0)} = \frac{\mathcal{Q}}{\mathcal{H}} e^{-\Phi} , \tag{7.5}$$

$$\widetilde{C}_{(2)} = C_{(2)} + \frac{\mathcal{Q}}{\mathcal{H}} e^{-\Phi} \frac{\eta r^6 L^2 \mathcal{H}^{1/2}}{r^4 + f_4' \eta^2 L^2 \mathcal{H}^{1/2} r_+^4} (F dt + dy) \wedge (f_4' d\chi + f_6' d\alpha_3 + f_7' d\alpha_2) , \tag{7.6}$$

$$\begin{aligned}\widetilde{C}_{(4)} = & \; C_{(4)} + C_{(2)} \wedge \frac{\eta r^6 L^2 \mathcal{H}^{1/2}}{r^4 + f_4' \eta^2 L^2 \mathcal{H}^{1/2} r_+^4} (F dt + dy) \wedge (f_4' d\chi + f_6' d\alpha_3 + f_7' d\alpha_2) \\ & + \frac{\eta L^4 r^4}{4} \Bigg[b_0' d\mu + b_1' d\alpha_1 + b_2' d\alpha_2 + b_3' d\alpha_3 + 3\sigma L^4 \left(w_1 d\mu + \frac{1}{2} w_2 d\alpha_1 \right) \\ & + \gamma L^4 \mathcal{G} \left(\frac{1}{2} \mu_1^2 (\mu_2^2 - \mu_3^2) d\alpha_3 + \left(\frac{1}{2} \mu_1^2 \mu_2^2 + \frac{1}{2} \mu_1^2 \mu_3^2 - \mu_2^2 \mu_3^2 \right) d\alpha_2 \right) \Bigg] \wedge (dt + dy) \wedge dx_1 \wedge dx_2 .\end{aligned} \tag{7.7}$$

In the above formulae we have defined

$$F(r) = 1 - \frac{r_+^4}{r^4} , \qquad \kappa(r) = \mathcal{H}^{1/2} L^2 \eta^2 \frac{r_+^4}{r^2} . \tag{7.8}$$

For $\beta = 0$ the background reduces to the null Melvin twist of non-extremal D3 brane solution along the Hopf fiber found in [8, 10] and presented in section 2.

By putting $r_+ = 0$ we get the non-relativistic zero temperature Lunin-Maldacena solution. As advertised above, it is invariant under the Schrödinger symmetry and has dynamical exponent $\nu = 2$. The solution has non-zero NS and RR internal fluxes and the characteristic rotation-like components of the metric generated by the null Melvin twist.

A further generalization of the solution presented in this section is possible. One can take the most general Lunin-Maldacena solution with three deformation parameters [41, 51] and apply to it the general three-parameter null Melvin twist from section 2. This will generate a six-parameter deformation of $AdS_5 \times S^5$ and the resulting background should be invariant under the Schrödinger symmetry.

8 Comments on the dual field theory

Although we have not studied the non-relativistic theories dual to the gravity solutions constructed above in much detail, we would like to offer some comments.

First of all let us review the large N_c gauge theories dual to the gravity solutions before we apply the null Melvin twist. The $\mathcal{N} = 4$ SYM has three adjoint chiral superfields, denoted by Φ_I, where $I = 1, 2, 3$. It is possible to consider a mass perturbation to the superpotential of $\mathcal{N} = 4$ SYM of the form

$$\Delta W \sim m_1 \Phi_1^2 + m_2 \Phi_2^2 . \tag{8.1}$$

This is a relevant operator and therefore triggers an RG flow.

The case $m_2 = 0$ (or $m_1 = 0$) was studied by Leigh and Strassler in [30] who showed that after integrating the massive chiral superfield the theory flows to an IR fixed point with

$\mathcal{N}=1$ supersymmetry and $\mathrm{SU}(2)\times\mathrm{U}(1)_R$ global symmetry. The type IIB gravity dual of this RG flow was constructed in [34] and we applied the null Melvin twist to it in section 4.

The more general case when both m_1 and m_2 are non-zero and equal to each other an $\mathcal{N}=2^*$ supersymmetric RG flow with $\mathrm{SU}(2)_R\times\mathrm{U}(1)$ global symmetry is triggered. The theory flows to the Coulomb branch and does not have a fixed point in the IR [52]. The gravity solution dual to this mass deformation was found in [36] and we discussed its null Melvin twist in section 6.

In the most general case, one can consider turning on masses for all three chiral superfields. This theory has $\mathcal{N}=1$ supersymmetry and again undergoes an RG flow resulting in a very rich structure in the IR [53]. However its exact ten-dimensional gravity dual is not known so we cannot apply the null Melvin twist to it.[11]

$\mathcal{N}=4$ SYM has also a particular set of marginal deformations parametrized by a complex parameter β and therefore known as the β-deformations [30]. In this case, the superpotential of $\mathcal{N}=4$ SYM is modified in the following way

$$W\sim Tr(\Phi_1[\Phi_2,\Phi_3]) \quad \rightarrow \quad W_\beta\sim Tr(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3-e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2)\,. \tag{8.2}$$

The field theories described by this superpotential are conformal and have $\mathcal{N}=1$ supersymmetry. Their gravity duals were constructed in [31] by a procedure very similar to the null Melvin twist. One starts with $\mathrm{AdS}_5\times S^5$ and performs an S-duality followed by a TsT transformation on two isometry directions in the internal manifold and then one more S-duality. This construction can be easily extended to the non-extremal $D3$ brane solution [50] which corresponds to turning on temperature in the dual field theory. We discussed the non-relativistic version of the Lunin-Maldacena solution in section 7.

Finally let us comment on the relativistic field theory dual the family of RG flows and fixed points discussed in section 5. We begin with $\mathbb{Z}_2$ quiver gauge theory with an $\mathrm{SU}(N)\times\mathrm{SU}(N)$ gauge group, two hypermultiplets (A_1,B_2) and (B_1,A_2) and a pair of adjoint chiral superfields (Φ_1,Φ_2) [38, 54]. The theory is conformal and has $\mathcal{N}=2$ supersymmetry. It can be deformed by the following mass term

$$\Delta W\sim m_1\Phi_1^2+m_2\Phi_2^2\,. \tag{8.3}$$

In the infrared one can integrate this mass term and find a family of $\mathcal{N}=1$ conformal fixed points parametrized by the ratio m_1/m_2. Since the m_i's are in general complex numbers this is actually a $\mathbb{CP}^1$ worth of conformal IR fixed points [55]. The explicit gravity duals of this family of fixed points are known, for $m_1=m_2$ one gets the Pilch-Warner fixed point [34] and for $m_1=-m_2$ one finds the Klebanov-Witten fixed point (see figure 1). The gravity solutions for an arbitrary value of m_1/m_2 interpolating between these two fixed points were constructed in [39] (see also [56]).

By performing the null Melvin twist we introduce a particular deformation of the type IIB background which is proportional to the real parameter η. When we put $\eta=0$ in our gravity solutions we recover the Discrete Light-Cone Quantization (DLCQ) of the original solutions (without the null Melvin twist) along the compact light-like direction $v\sim(t-y)$.

[11] Linearized solutions of type IIB were found in [53].

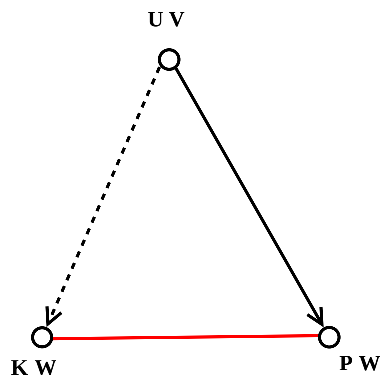

Figure 1. The massive RG flow from a $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM in the UV to the KW and PW fixed points in the IR. The horizontal (red) line represents a family of IR fixed points interpolating between the KW and PW solutions. The solid lines indicate that the supergravity solutions are known (at least numerically). The structure of this RG flows and the family of fixed points is the same after the null Melvin twist, in particular there is a line of fixed point solutions with Schrödinger symmetry.

The DLCQ amounts to requiring all fields to be invariant under translations along the light-like direction v. The way to include the deformation introduced by the null Melvin twist was discussed in [10, 57]. One simply has to twist the momentum generator J_v by the momentum generator along the isometry direction used for the null Melvin twist[12]

$$\tilde{J}_v = J_v - \eta J_\phi \ . \tag{8.4}$$

So the deformed DLCQ that we have to perform for $\eta \neq 0$ amounts to requiring all fields to be invariant under shifts generated by $\tilde{J}_v$. As noted in [10], $\tilde{J}_v$ is not light-like for $\eta \neq 0$, therefore one technically has a DLCQ only from the point of view of the boundary field theory. Although our gravity solutions look quite messy the discussion above apply to all of them which have the Schrödinger symmetry.

The meaning of this deformed DLCQ in the dual field theory is well understood [10, 57]. One has to perform a deformed DLCQ with the twisted generator $\tilde{J}_v$ on the undeformed theory dual to the gravity solution before the null Melvin twist, now J_ϕ has to be substituted with its dual R-symmetry generator. The momenta in the v direction of all fields get shifted by a certain amount proportional to η[13]

$$P_v = \frac{N}{R_v} + \eta Q \,, \tag{8.5}$$

where N is an integer, Q is the R-charge of the field under the U(1) R-symmetry that we use for the null Melvin twist and R_v is the radius of the compact v direction, $v \sim v + 2\pi R_v$.

[12]For the general Melvin twist of section 2 we have $\tilde{J}_v = J_v - \sum_{i=1}^{3} \eta_i J_{\varphi_i}$

[13]For the general null Melvin twist we have to modify this to $P_v = \dfrac{N}{R_v} + \displaystyle\sum_{i=1}^{3} \eta_i Q_i$, where Q_i are the R-charges under $\mathrm{U}(1)_{\phi_i}$.

JHEP07(2009)098

Alternatively, one can think of the field theory dual to the gravitational solution after the null Melvin twist as a theory in which the products of fields get modified in the following way[14]

$$\Psi_1\Psi_2 \quad \to \quad \Psi_1 \star \Psi_2 = e^{2\pi i\eta(P_v^1 Q^2 - P_v^2 Q^1)}\Psi_1\Psi_2\,, \tag{8.6}$$

where P_v^i is the momentum of the field Ψ_i along v and Q_R^i is its R-charge along the U(1) direction used for the null Melvin twist.

We can conclude that the field theory duals of the gravity solutions that we obtained via the null Melvin twist can be obtained by applying the deformed DLCQ procedure described above to the field theory dual to the gravity solution before the null Melvin twist. In particular the solution discussed in section 4 is dual to the deformed DLCQ of a mass deformation of $\mathcal{N} = 4$ SYM where one gives mass to one of the chiral superfields. We find that the gravity dual suggests, much like its parent relativistic theory, that the non-relativistic theory flows to an IR fixed point after the massive field is integrated out, at this fixed point the field theory has dynamical exponent $\nu = 2$ and is invariant under the Schrödinger symmetry. The solution in section 7 is dual to a deformed DLCQ of the β-deformation of $\mathcal{N} = 4$ SYM discussed in [30, 31], the non-relativistic field theory is again Schrödinger invariant. The solution of section 6 is a bit more subtle since it does not flow to an IR fixed point. One can think of it in the following way - perform a deformed DLCQ of $\mathcal{N} = 4$ SYM and then give equal masses to two of the chiral superfields. The theory undergoes an RG flow similar to the RG flow of its parent relativistic theory, there is no IR fixed point and the theory flows to a Coulomb branch. Finally the solutions in section 5 should be dual to a family of non-relativistic IR fixed points with Schrödinger symmetry, which can be obtained by a mass deformation of the deformed DLCQ of the $\mathcal{N} = 2$ quiver gauge theory obtained by a $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM. Note that in all cases we considered we have at least U(1) R-symmetry and we used exactly the corresponding isometry direction in the internal manifold for the null Melvin twist. However in all cases the U(1) R-symmetry is a different U(1) subgroup of the SO(6) R-symmetry of $\mathcal{N} = 4$ SYM which will lead to a different definition of the star product (8.6). Note also that since we are discussing supergravity solutions, strictly speaking, they should be considered as duals to field theories at large N_c.

Let us also comment on the meaning of the parameter η. As pointed out in refs. [8, 10], the particle number density in the non-relativistic field theory is proportional to some power of η/R_v. However for the extremal gravity solutions (i.e. $T = 0$) this parameter can be set to unity by rescaling $u \to \eta u$, $v \to \eta^{-1} v$. Since the RG flow solutions considered here are only known for $T = 0$ we can remove the dependence of the metric on η, however we chose to keep η explicit to emphasize that it has a physical meaning and cannot be removed at finite temperature. We found the finite temperature non-relativistic Lunin-Maldacena solution and thus the dual field theory has two important parameters: the temperature T and the particle number density which is related to η. It will be interesting to study the thermodynamics of this theory and see if the marginal deformation parameter β affects

[14]These theories are called dipole theories and the modified product is called a star product, see [57–59] for a more detailed discussion of this kind of field theories and their gravity duals.

JHEP07(2009)098

any of the interesting thermodynamical quantities. For this finite temperature solution we can of course take the limit $\eta \to 0\,, T =$ fixed which will lead to the trivial case of zero particle number density in the dual field theory.

9 Conclusions

In this note we found the non-relativistic generalizations of some known gravity solutions which are dual to RG flows and marginal deformations of $\mathcal{N} = 4$ SYM. We found that when the original type IIB background dual to the relativistic theory has an AdS_5 fixed point the non-relativistic geometry obtained after the null Melvin twist has the Schrödinger symmetry. This is quite a generic feature and it should persist for other AdS compactifications with internal U(1) isometries to which one can apply the null Melvin twist.

Another general feature of all backgrounds that we discussed is the existence of RR and NS internal fluxes. In particular after the null Melvin twist the internal NS flux translates into off-diagonal, rotation-like, terms in the metric. More generally it will be interesting to understand if there is a consistent truncation of Schrödinger invariant ten dimensional IIB backgrounds with internal fluxes to five dimensions and if one can realize holographic RG flows directly in the five-dimensional gravitational theory.

It will be also quite interesting to look at the Polchinski-Strassler solution [53] and see how much of its rich structure is present in its non-relativistic version. The exact gravity solution dual to this theory is not known so one cannot straightforwardly apply the null Melvin twist. There are certainly other known solutions of type IIB and eleven-dimensional supergravity which are dual to RG flows of relativistic gauge theories [46, 60]. It will be interesting to find their non-relativistic generalizations and understand the dual field theories. In particular the eleven-dimensional solution of [46] realizes a holographic RG flow between $AdS_4 \times S^7$ in the UV and an AdS_4 compactification with fluxes in the IR, which is the eleven-dimensional analog of the solution in [34]. It should be possible to find the non-relativistic version of this solution which should be dual to a non-relativistic conformal field theory in $1 + 1$ dimensions.

Since one of the goals of extending gauge/gravity duality to non-relativistic field theories is to be able to gain some understanding of these theories at strong coupling and finite temperature, it will be quite interesting to construct the finite temperature cousins of the non-relativistic RG flows that were discussed here.[15] This will be a rather non-trivial task and maybe one has to look for approximate solutions and extract the interesting physics from them.

The general null Melvin twist discussed in section 2 relies on the $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$ subgroup of the SO(6) isometry group of S^5. One can use this to generate a large class of non-relativistic type IIB backgrounds by applying the null Melvin twist to solutions of the form $\mathrm{AdS}_5 \times L^{p,q,r}$ where $L^{p,q,r}$ are the five-dimensional Sasaki-Einstein manifolds constructed in [61]. These manifolds have $\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)$ isometry and are a generalization of the $Y^{p,q}$ Sasaki-Einstein manifolds found in [62]. For generic values of (η_1, η_2, η_3) we expect that these non-relativistic solutions will break supersymmetry completely, but there might be special choices of (η_1, η_2, η_3) for which some Killing spinors are preserved [42].

[15] See [40] for some work on the $\mathcal{N} = 2^*$ PW RG flow at finite temperature.

JHEP07(2009)098

This note can be considered as a modest first step in constructing holographic duals of non-relativistic CFT's deformed by relevant or marginal operators, however it can be a hint for the construction of a more general ansatz for other gravity solutions dual to non-relativistic CFT's. These may include solutions with different amounts of supersymmetry [42], as well as solutions with different dynamical exponents and brane wave deformations [15]. We hope to return to some of these problems in the near future.

Acknowledgments

We would like to thank Nick Halmagyi, Mike Mulligan and Kentaroh Yoshida for useful discussions and comments. We are grateful to Mukund Rangamani for numerous illuminating conversations and explanations. We would also like to thank Clifford Johnson, Krzysztof Pilch and Nick Warner for their interest in this work, many useful discussions and for the constant encouragement. We would especially like to thank Clifford Johnson for valuable comments on the manuscript. This work is supported in part by the DOE grant DE-FG03-84ER-40168. NB is supported also by a Graduate Fellowship from KITP and in part by the National Science Foundation under Grant No. PHY05-51164.

A Null Melvin twist of the Pilch-Warner flow

Here we present in detail the null Melvin twist of the $\mathcal{N}=1$ PW RG flow solution. It is convenient to rewrite the solution in a more compact form. The metric is (we put $y \equiv x_3$)

$$\begin{aligned} ds_{10}^2 &= \Omega^2 e^{2A}(-dt^2 + dy^2 + dx_1^2 + dx_2^2) + \Omega^2 dr^2 + \frac{L^2\Omega^2}{\rho^2\cosh^2\chi} d\theta^2 \\ &\quad + f_1\, d\alpha_1^2 + f_2\, d\alpha_2^2 + f_3\, d\alpha_3^2 + f_4\, d\phi^2 + 2f_5\, d\alpha_2 d\alpha_3 + 2f_6\, d\alpha_3 d\phi + 2f_7\, d\alpha_2 d\phi\,. \end{aligned} \tag{A.1}$$

The NS and RR two-form potentials are

$$\begin{aligned} B_{(2)} &= b_1\, d\phi\wedge d\alpha_1 + b_2\, d\alpha_3\wedge d\alpha_1 + b_3\, d\alpha_2\wedge d\alpha_1 + b_4\, d\theta\wedge d\alpha_2 + b_5\, d\theta\wedge d\alpha_1 \\ &\quad + b_6\, d\phi\wedge d\alpha_2 + b_7\, d\alpha_3\wedge d\alpha_2\,. \end{aligned} \tag{A.2}$$

$$\begin{aligned} C_{(2)} &= c_1\, d\phi\wedge d\alpha_1 + c_2\, d\alpha_3\wedge d\alpha_1 + c_3\, d\alpha_2\wedge d\alpha_1 + c_4\, d\theta\wedge d\alpha_2 + c_5\, d\theta\wedge d\alpha_1 \\ &\quad + c_6\, d\phi\wedge d\alpha_2 + + c_7\, d\alpha_3\wedge d\alpha_2\,. \end{aligned} \tag{A.3}$$

Where we have defined

$$\begin{aligned} f_1 &= \frac{L^2\rho^2}{\Omega^2}\frac{\cos^2\theta}{4}\,, \\ f_2 &= \frac{L^2\rho^2}{\Omega^2}\frac{\cos^2\theta}{4}\sin^2\alpha_1 + L^2\Omega^2\cos^2\alpha_1\left[\frac{\rho^4}{X_1^2}\frac{\cos^4\theta}{4} + \frac{\sin^2\theta\cos^2\theta}{4\rho^2\cosh^2\chi}\left(1 - \frac{1-\rho^6}{X_1}\cos^2\theta\right)^2\right], \\ f_3 &= L^2\Omega^2\left[\frac{\rho^4}{X_1^2}\frac{\cos^4\theta}{4} + \frac{\sin^2\theta\cos^2\theta}{4\rho^2\cosh^2\chi}\left(1 - \frac{1-\rho^6}{X_1}\cos^2\theta\right)^2\right], \\ f_4 &= \frac{L^2\Omega^2\rho^4}{X_1^2}\left(1 - \frac{3}{2}\cos^2\theta\right)^2 + \frac{L^2\Omega^2}{\rho^2\cosh^2\chi}\sin^2\theta\cos^2\theta\left(\frac{3}{2} + \frac{1-\rho^6}{X_1}\left(1 - \frac{3}{2}\cos^2\theta\right)\right)^2, \end{aligned} \tag{A.4}$$

$$
\begin{aligned}
f_5 &= f_3 \cos\alpha_1\,,\\
f_6 &= \frac{L^2\Omega^2\rho^4}{X_1^2}\frac{\cos^2\theta}{2}\left(1-\frac{3}{2}\cos^2\theta\right)\\
&\quad -\frac{L^2\Omega^2}{\rho^2\cosh^2\chi}\frac{\sin^2\theta\cos^2\theta}{2}\left(1-\frac{1-\rho^6}{X_1}\cos^2\theta\right)\left(\frac{3}{2}+\frac{1-\rho^6}{X_1}\left(\sin^2\theta-\frac{1}{2}\cos^2\theta\right)\right)\,,\\
f_7 &= f_6\cos\alpha_1\,,
\end{aligned}
$$

and

$$
\begin{aligned}
b_1 &= \frac{L^2}{4}\tanh\chi\cos^2\theta\sin\theta\cos\alpha_3\left(\frac{3}{2}+\frac{1-\rho^6}{X_1}\left(1-\frac{3}{2}\cos^2\theta\right)\right)\,,\\
b_2 &= \frac{L^2}{4}\tanh\chi\cos^2\theta\sin\theta\cos\alpha_3\left(\frac{1-\rho^6}{2X_1}\cos^2\theta-\frac{1}{2}\right)\,, \qquad b_3 = b_2\cos\alpha_1\,, \qquad\qquad (A.5)\\
b_4 &= -\frac{L^2}{4}\tanh\chi\cos\theta\cos\alpha_3\sin\alpha_1\,, \qquad b_5 = \frac{L^2}{4}\tanh\chi\cos\theta\sin\alpha_3\,,\\
b_6 &= b_1\tan\alpha_3\sin\alpha_1\,, \qquad b_7 = b_2\tan\alpha_3\sin\alpha_1\,,
\end{aligned}
$$

and

$$
\begin{aligned}
&c_1 = -b_6\frac{1}{\sin\alpha_1} = -b_1\,\tan\alpha_3\,, \qquad c_2 = -b_7\frac{1}{\sin\alpha_1} = -b_2\,\tan\alpha_3\,,\\
&c_3 = -b_7\,\cot\alpha_1 = -b_3\,\tan\alpha_3\,, \qquad c_4 = -b_4\,\tan\alpha_3\,, \qquad\qquad (A.6)\\
&c_5 = -b_4\frac{1}{\sin\alpha_1} = b_5\cot\alpha_3\,, \qquad c_6 = b_1\,\sin\alpha_1 = b_6\cot\alpha_3\,, \qquad c_7 = b_2\,\sin\alpha_1 = b_7\cot\alpha_3\,.
\end{aligned}
$$

Now we are ready to the apply the five steps of the null Melvin twist.

Step 1. Perform a boost in the (t, y) plane with a parameter γ_0

$$
t \to ct - sy\,, \qquad y \to -st + cy\,, \qquad (A.7)
$$

where

$$
c = \cosh\gamma_0\,, \qquad s = \sinh\gamma_0\,, \qquad (A.8)
$$

note that $c^2 - s^2 = 1$. The whole background is invariant under this boost so nothing is changed.

Step 2. Perform T-duality along y. To avoid clutter we will not show the explicit changes in the RR-forms at each steps, but present the final result.

At this step the B-field is invariant under this T-duality, the only changes are in the metric and the dilaton which take the form

$$
\begin{aligned}
ds_{10}^2 &= \frac{1}{\Omega^2 e^{2A}}dy^2 + \Omega^2 e^{2A}(-dt^2+dx_1^2+dx_2^2) + \Omega^2 dr^2 + \frac{L^2\Omega^2}{\rho^2\cosh^2\chi}d\theta^2 \qquad (A.9)\\
&\quad + f_1\,d\alpha_1^2 + f_2\,d\alpha_2^2 + f_3\,d\alpha_3^2 + f_4\,d\phi^2 + 2f_5\,d\alpha_2 d\alpha_3 + 2f_6\,d\alpha_3 d\phi + 2f_7\,d\alpha_2 d\phi\,,\\
\widetilde{\Phi} &= \Phi - \log(\Omega e^A)\,. \qquad (A.10)
\end{aligned}
$$

Step 3. Perform a shift $\phi \to \phi + ay$. The dilaton is invariant under this shift, but the metric and the B-field change to

$$\begin{aligned} ds^2_{10} &= \left(\frac{1}{\Omega^2 e^{2A}} + a^2 f_4\right) dy^2 + 2af_4 dy d\phi + 2af_6 dy d\alpha_3 + 2af_7 dy d\alpha_2 \\ &\quad + \Omega^2 e^{2A}(-dt^2 + dx_1^2 + dx_2^2) + \Omega^2 dr^2 + \frac{L^2\Omega^2}{\rho^2 \cosh^2\chi} d\theta^2 + f_1\, d\alpha_1^2 + f_2\, d\alpha_2^2 \\ &\quad + f_3\, d\alpha_3^2 + f_4 d\phi^2 + 2f_5\, d\alpha_2 d\alpha_3 + 2f_6\, d\alpha_3 d\phi + 2f_7\, d\alpha_2 d\phi\,, \qquad \text{(A.11)} \\ B_{(2)} &= ab_1\, dy \wedge d\alpha_1 + ab_6\, dy \wedge d\alpha_2 + b_1\, d\phi \wedge d\alpha_1 + b_2\, d\alpha_3 \wedge d\alpha_1 + b_3\, d\alpha_2 \wedge d\alpha_1 \\ &\quad + b_4\, d\theta \wedge d\alpha_2 + b_5\, d\theta \wedge d\alpha_1 + b_6\, d\phi \wedge d\alpha_2 + b_7\, d\alpha_3 \wedge d\alpha_2\,. \qquad \text{(A.12)} \end{aligned}$$

Step 4. Perform one more T-duality along y. All NS fields change (not all components change though) under this transformation and the end result is

$$\begin{aligned} ds^2_{10} &= h_1 dy^2 - 2ab_1 h_1 dy d\alpha_1 - 2ab_6 h_1 dy d\alpha_2 + \Omega^2 e^{2A}(-dt^2 + dx_1^2 + dx_2^2) + \Omega^2 dr^2 \\ &\quad + \frac{L^2\Omega^2}{\rho^2\cosh^2\chi} d\theta^2 + (f_1 + a^2 b_1^2 h_1)\, d\alpha_1^2 + (f_2 + a^2(b_6^2 - f_7^2)h_1)\, d\alpha_2^2 + (f_3 - a^2 f_6^2 h_1)\, d\alpha_3^2 \\ &\quad + \frac{f_4}{1 + a^2 f_4 \Omega^2 e^{2A}}\, d\phi^2 + 2a^2 b_1 b_6 h_1 d\alpha_1 d\alpha_2 + 2(f_5 - a^2 f_6 f_7 h_1)\, d\alpha_2 d\alpha_3 \\ &\quad + \frac{2f_6}{1 + a^2 f_4 \Omega^2 e^{2A}}\, d\alpha_3 d\phi + \frac{2f_7}{1 + a^2 f_4 \Omega^2 e^{2A}}\, d\alpha_2 d\phi\,, \qquad \text{(A.13)} \\ B_{(2)} &= af_4 h_1\, d\phi \wedge dy + af_7 h_1\, d\alpha_2 \wedge dy + af_6 h_1\, d\alpha_3 \wedge dy + (b_1 - a^2 h_1 b_1 f_4)\, d\phi \wedge d\alpha_1 \\ &\quad + (b_2 - a^2 h_1 b_1 f_6)\, d\alpha_3 \wedge d\alpha_1 + (b_3 - a^2 h_1 b_1 f_7)\, d\alpha_2 \wedge d\alpha_1 + b_4\, d\theta \wedge d\alpha_2 \\ &\quad + b_5\, d\theta \wedge d\alpha_1 + (b_6 - a^2 h_1 b_6 f_4)\, d\phi \wedge d\alpha_2 + (b_7 - a^2 h_1 b_6 f_6)\, d\alpha_3 \wedge d\alpha_2\,. \qquad \text{(A.14)} \\ \widetilde{\Phi} &= \Phi - \log\left(1 + a^2 f_4 \Omega^2 e^{2A}\right)\,, \qquad \text{(A.15)} \end{aligned}$$

where we have defined

$$h_1 = \frac{\Omega^2 e^{2A}}{1 + a^2 f_4 \Omega^2 e^{2A}}\,, \qquad \text{(A.16)}$$

Step 5. Perform one more boost in the (t, y) plane with a parameter $-\gamma_0$

$$t \to ct + sy\,, \qquad y \to st + cy\,, \qquad \text{(A.17)}$$

then take the following limit

$$a \to 0\,, \qquad \gamma_0 \to \infty\,, \qquad as = ac = \eta\,. \qquad \text{(A.18)}$$

The final form of the metric is

$$\begin{aligned} ds^2_{10} &= -f_4 \Omega^4 e^{4A}[\eta(dt + dy)]^2 - \Omega^2 e^{2A}(dt - dy)(dt + dy) - 2b_1 \Omega^2 e^{2A}[\eta(dt + dy)]d\alpha_1 \\ &\quad - 2b_6 \Omega^2 e^{2A}[\eta(dt + dy)]d\alpha_2 + \Omega^2 e^{2A}(dx_1^2 + dx_2^2) + \Omega^2 dr^2 + \frac{L^2\Omega^2}{\rho^2\cosh^2\chi} d\theta^2 + f_1\, d\alpha_1^2 \\ &\quad + f_2\, d\alpha_2^2 + f_3\, d\alpha_3^2 + f_4\, d\phi^2 + 2f_5\, d\alpha_2 d\alpha_3 + 2f_6\, d\alpha_3 d\phi + 2f_7\, d\alpha_2 d\phi\,. \qquad \text{(A.19)} \end{aligned}$$

The B-field and the dilaton are

$$\begin{aligned}
B_{(2)} &= f_4\Omega^2 e^{2A} d\phi\wedge[\eta(dt+dy)] + f_7\Omega^2 e^{2A} d\alpha_2\wedge[\eta(dt+dy)] + f_6\Omega^2 e^{2A} d\alpha_3\wedge[\eta(dt+dy)] \\
&\quad + b_1\ d\phi\wedge d\alpha_1 + b_2\ d\alpha_3\wedge d\alpha_1 + b_3\ d\alpha_2\wedge d\alpha_1 + b_4\ d\theta\wedge d\alpha_2 + b_5\ d\theta\wedge d\alpha_1 \\
&\quad + b_6\ d\phi\wedge d\alpha_2 + b_7\ d\alpha_3\wedge d\alpha_2\,, \\
\widetilde{\Phi} &= \Phi - \log\left(1 + a^2 f_4\Omega^2 e^{2A}\right) \to \Phi\,. \qquad (A.20)
\end{aligned}$$

Note that the metric on the compact manifold is the same after the Melvin twist. Note also that there are off-diagonal metric coefficients $g_{u\alpha_1}$ and $g_{u\alpha_2}$ where $u = (t+y)$. These metric coefficients vanish at the UV fixed point because $b_1 \sim b_6 \sim \sinh\chi$ and $\chi = 0$. However at the IR fixed point the off-diagonal metric coefficients are not vanishing.

In the RR-sector, $G_{(3)} = dC_{(2)}$ remains invariant, but $F_{(5)}$ changes under the Melvin twist procedure. Here we present the final form of the five-form flux

$$\begin{aligned}
\widetilde{F}_{(5)} &= (1+\star)[dx^0\wedge dx^1\wedge dx^2\wedge dy\wedge(w_r(r,\theta)dr + w_\theta(r,\theta)d\theta) \\
&\quad + \left(\Omega^2 e^{2A}\right) du\wedge(f_4 d\phi + f_6 d\alpha_3 + f_7 d\alpha_2)\wedge G_{(3)}]\,. \qquad (A.21)
\end{aligned}$$

B Family of fixed points

This appendix is devoted to a short review of the solution found in [39]. The metric is (note that the σ_i in [39] are the same as our σ_i)

$$\begin{aligned}
ds^2_{10} &= \hat{\Omega}^2 e^{2A}(-dt^2 + dy^2 + dx_1^2 + dx_2^2) + \hat{\Omega}^2 dr^2 + \hat{f}_8 d\theta^2 + \hat{f}_1\ d\alpha_1^2 + 2\hat{f}_9\ d\alpha_1 d\alpha_2 \\
&\quad + \hat{f}_2\ d\alpha_2^2 + \hat{f}_3\ d\alpha_3^2 + \hat{f}_4\ d\phi^2 + 2\hat{f}_5\ d\alpha_2 d\alpha_3 + 2\hat{f}_6\ d\alpha_3 d\phi + 2\hat{f}_7\ d\alpha_2 d\phi\,. \qquad (B.1)
\end{aligned}$$

The NS and RR two-form potentials are

$$\begin{aligned}
B_{(2)} &= \hat{b}_1\ d\phi\wedge d\alpha_1 + \hat{b}_2\ d\alpha_3\wedge d\alpha_1 + \hat{b}_3\ d\alpha_2\wedge d\alpha_1 + \hat{b}_4\ d\theta\wedge d\alpha_2 + \hat{b}_5\ d\theta\wedge d\alpha_1 \\
&\quad + \hat{b}_6\ d\phi\wedge d\alpha_2 + \hat{b}_7\ d\alpha_3\wedge d\alpha_2\,, \qquad (B.2) \\
C_{(2)} &= \hat{c}_1\ d\phi\wedge d\alpha_1 + \hat{c}_2\ d\alpha_3\wedge d\alpha_1 + \hat{c}_3\ d\alpha_2\wedge d\alpha_1 + \hat{c}_4\ d\theta\wedge d\alpha_2 + \hat{c}_5\ d\theta\wedge d\alpha_1 \\
&\quad + \hat{c}_6\ d\phi\wedge d\alpha_2 + +\hat{c}_7\ d\alpha_3\wedge d\alpha_2\,. \qquad (B.3)
\end{aligned}$$

There is also a non-trivial dilaton, $\Phi(\theta)$, for the family of fixed points. It vanishes at the PW and KW fixed points. Above we have defined

$$\begin{aligned}
\hat{f}_1 &= L_{IR}^2\hat{\Omega}^{-2}(A_1^2\cos^2\alpha_3 + A_2^2\sin^2\alpha_3)\,, \\
\hat{f}_2 &= L_{IR}^2\hat{\Omega}^{-2}\left[(A_1^2\sin^2\alpha_3 + A_2^2\cos^2\alpha_3)\sin^2\alpha_1 + A_3^2\cos^2\alpha_1\right]\,, \\
\hat{f}_3 &= L_{IR}^2\hat{\Omega}^{-2}A_3^2\,, \qquad \hat{f}_4 = L_{IR}^2\hat{\Omega}^{-2}(A_5^2 + A_3^2B_1^2)\,, \qquad \hat{f}_5 = \hat{f}_3\cos\alpha_1\,, \qquad (B.4) \\
\hat{f}_6 &= L_{IR}^2\hat{\Omega}^{-2}A_3^2B_1\,, \qquad \hat{f}_7 = \hat{f}_6\cos\alpha_1\,, \\
\hat{f}_8 &= L_{IR}^2\hat{\Omega}^{-2}A_4^2\,, \qquad \hat{f}_9 = L_{IR}^2\hat{\Omega}^{-2}(A_1^2 - A_2^2)\sin\alpha_3\cos\alpha_3\sin\alpha_1\,, \qquad (B.5) \\
\hat{\Omega}^2 &= -\frac{3}{2}A_3A_4A_5\frac{1}{(A_1A_2)'}\,,
\end{aligned}$$

where $'$ denotes taking derivative with respect to θ and

$$\hat{b}_1 = -L_{IR}^2 F_2 \cos\alpha_3\,, \quad \hat{b}_2 = -L_{IR}^2 F_1 \cos\alpha_3\,, \quad \hat{b}_3 = \hat{b}_2 \cos\alpha_1\,, \quad \hat{b}_4 = L_{IR}^2 F_3 \cos\alpha_3 \sin\alpha_1\,,$$
$$\hat{b}_5 = -L_{IR}^2 F_3 \sin\alpha_3\,, \quad \hat{b}_6 = \hat{b}_1 \tan\alpha_3 \sin\alpha_1\,, \quad \hat{b}_7 = \hat{b}_2 \tan\alpha_3 \sin\alpha_1\,. \tag{B.6}$$

Also

$$\hat{c}_1 = -L_{IR}^2 H_2 \sin\alpha_3\,, \quad \hat{c}_2 = -L_{IR}^2 H_1 \sin\alpha_3\,, \quad \hat{c}_3 = \hat{c}_2 \cos\alpha_1\,, \quad \hat{c}_4 = -L_{IR}^2 H_3 \sin\alpha_3 \sin\alpha_1\,,$$
$$\hat{c}_5 = -L_{IR}^2 H_3 \cos\alpha_3\,, \quad \hat{c}_6 = -\hat{c}_1 \cot\alpha_3 \sin\alpha_1\,, \quad \hat{c}_7 = \hat{c}_2 \cot\alpha_3 \sin\alpha_1\,. \tag{B.7}$$

The functions A_i and B_1 depend only ont θ and are defined in [39]. We have introduced six new functions of θ – F_i and H_i. They are related to the functions used in [39] as follows. The three-from fluxes in [39] are

$$\begin{aligned} H_{(3)} = dB_{(2)} = (g_1+g_4)\frac{A_1A_3A_4}{\hat{\Omega}^3}\,\sigma_1\wedge\sigma_3\wedge d\theta + \frac{A_1A_4}{\hat{\Omega}^3}\left[(g_1+g_4)A_3B_1 - (g_2+g_5)A_5\right]\sigma_1\wedge d\phi\wedge d\theta \\ +(-ig_3+ig_6)\frac{A_2A_3A_5}{\hat{\Omega}^3}\,\sigma_2\wedge\sigma_3\wedge d\phi\,, \end{aligned} \tag{B.8}$$

$$\begin{aligned} G_{(3)} = dC_{(2)} = (g_4-g_1)\frac{A_2A_3A_4}{\hat{\Omega}^3}\,\sigma_2\wedge\sigma_3\wedge d\theta + \frac{A_2A_4}{\hat{\Omega}^3}\left[(g_4-g_1)A_3B_1 - (g_5-g_2)A_5\right]\sigma_2\wedge d\phi\wedge d\theta \\ +(-ig_3-ig_6)\frac{A_1A_3A_5}{\hat{\Omega}^3}\,\sigma_1\wedge\sigma_3\wedge d\phi\,. \end{aligned} \tag{B.9}$$

One can show that these fluxes come from the following potentials

$$B_{(2)} = F_1(\theta)\;\sigma_1\wedge\sigma_3 + F_2(\theta)\sigma_1\wedge d\phi + F_3(\theta)\sigma_2\wedge d\theta\,, \tag{B.10}$$
$$C_{(2)} = H_1(\theta)\;\sigma_2\wedge\sigma_3 + H_2(\theta)\sigma_2\wedge d\phi + H_3(\theta)\sigma_1\wedge d\theta\,, \tag{B.11}$$

if the following identities hold (here $' = \frac{d}{d\theta}$)

$$F_1' - F_3 = (g_1+g_4)\frac{A_1A_3A_4}{\hat{\Omega}^3}\,, \qquad F_2' = \frac{A_1A_4}{\hat{\Omega}^3}\left[(g_1+g_4)A_3B_1 - (g_2+g_5)A_5\right]\,, \tag{B.12}$$
$$F_2 = (-ig_3+ig_6)\frac{A_2A_3A_5}{\hat{\Omega}^3}\,.$$
$$H_1' + H_3 = (g_4-g_1)\frac{A_2A_3A_4}{\hat{\Omega}^3}\,, \qquad H_2' = \frac{A_2A_4}{\hat{\Omega}^3}\left[(g_4-g_1)A_3B_1 - (g_5-g_2)A_5\right]\,, \tag{B.13}$$
$$H_2 = (ig_3+ig_6)\frac{A_1A_3A_5}{\hat{\Omega}^3}\,.$$

There is also the usual self-dual five-form flux

$$\begin{aligned} F_{(5)} &= f_0 dt\wedge dx^1\wedge dx^2\wedge dy\wedge dr \\ &\quad + f_0\hat{\Omega}^{-10}\left(A_1A_2A_3A_4A_5\right)\sigma_1\wedge\sigma_2\wedge(\sigma_3 + B_1 d\phi + B_2 d\theta)\wedge d\theta\wedge d\phi\,. \end{aligned} \tag{B.14}$$

C Lunin-Maldacena at finite temperature

Here we will review the finite temperature Lunin-Maldacena solution [31] found in [50]. The metric is

$$
\begin{aligned}
ds_\beta^2 &= \mathcal{H}^{1/2}L^2\left[-r^2\left(1-\frac{r_+^4}{r^4}\right)dt^2+r^2(dx_1^2+dx_2^2+dy^2)+\left(1-\frac{r_+^4}{r^4}\right)^{-1}\frac{dr^2}{r^2}\right]\\
&\quad+\mathcal{H}^{1/2}L^2\left[\sum_{i=1}^{3}(d\mu_i^2+\mathcal{G}\mu_i^2d\varphi_i^2)+\mathcal{G}|\beta|^2L^4\mu_1^2\mu_2^2\mu_3^2(d\varphi_1+d\varphi_2+d\varphi_3)^2\right]. \qquad \text{(C.1)}
\end{aligned}
$$

The NS and RR fields in the deformed solution are (we have set the dilaton in the undeformed $\mathrm{AdS}_5\times S^5$ solution to zero)

$$B_{(2)} = \gamma\mathcal{G}\mathcal{B}_2-\sigma\mathcal{A}_2\,, \qquad e^{2\Phi}=\mathcal{G}\mathcal{H}^2\,. \qquad \text{(C.2)}$$

$$C_{(0)} = \mathcal{H}^{-1}\mathcal{Q}\,, \qquad C_{(2)}=-\gamma\mathcal{A}_2-\sigma\mathcal{G}\mathcal{B}_2\,. \qquad \text{(C.3)}$$

$$C_{(4)} = \mathcal{A}_4-\gamma^2\mathcal{G}\mathcal{B}_2\wedge\mathcal{A}_2+\frac{1}{2}\gamma\sigma\mathcal{A}_2\wedge\mathcal{A}_2\,, \qquad C_{(6)}=B_{(2)}\wedge C_{(4)}\,. \qquad \text{(C.4)}$$

Where

$$\beta=\gamma-i\sigma\,, \qquad \mathcal{Q}=L^4\gamma\sigma(\mu_1^2\mu_2^2+\mu_1^2\mu_3^2+\mu_2^2\mu_3^2)\,, \qquad \text{(C.5)}$$

$$\mathcal{G}=\frac{1}{1+L^4|\beta|^2(\mu_1^2\mu_2^2+\mu_1^2\mu_3^2+\mu_2^2\mu_3^2)}\,, \qquad \mathcal{H}=1+L^4\sigma^2(\mu_1^2\mu_2^2+\mu_1^2\mu_3^2+\mu_2^2\mu_3^2). \qquad \text{(C.6)}$$

And we have defined the forms [50]

$$\mathcal{A}_1=L^2(\mu_2^2d\varphi_2-\mu_3^2d\varphi_3)\,, \qquad \mathcal{B}_1=L^2\left[-\mu_1^2d\varphi_1+\frac{\mu_2^2\mu_3^2}{\mu_2^2+\mu_3^2}(d\varphi_2+d\varphi_3)\right], \qquad \text{(C.7)}$$

$$\mathcal{A}_2=\mathcal{C}_1\wedge(d\varphi_1+d\varphi_2+d\varphi_3)\,, \qquad \mathcal{B}_2=L^4\left(\mu_1^2\mu_2^2d\varphi_1\wedge d\varphi_2+\mu_1^2\mu_3^2d\varphi_3\wedge d\varphi_1+\mu_2^2\mu_3^2d\varphi_2\wedge d\varphi_3\right)\,, \qquad \text{(C.8)}$$

$$\mathcal{A}_4=L^4\frac{r^4}{4}dt\wedge dx_1\wedge dx_2\wedge dy+\mathcal{C}_1\wedge d\varphi_1\wedge d\varphi_2\wedge d\varphi_3\,, \qquad \text{(C.9)}$$

where

$$d\mathcal{C}_1=L^4\sin^3\vartheta\cos\vartheta\sin\xi\cos\xi\; d\vartheta\wedge d\xi. \qquad \text{(C.10)}$$

To facilitate the null Melvin twist we will need the explicit form for the B-field

$$
\begin{aligned}
B_{(2)} &= \gamma\mathcal{G}L^2\left[\mu_1^2\mu_2^2d\varphi_1\wedge d\varphi_2+\mu_1^2\mu_3^2d\varphi_3\wedge d\varphi_1+\mu_2^2\mu_3^2d\varphi_2\wedge d\varphi_3\right]\\
&\quad-\sigma L^4(w_1\;d\theta+w_2\;d\xi)\wedge(d\varphi_1+d\varphi_2+d\varphi_3), \qquad \text{(C.11)}
\end{aligned}
$$

where w_i are defined as

$$\mathcal{C}_1=L^4(w_1(\vartheta,\xi)\;d\vartheta+w_2(\vartheta,\xi)\;d\xi)\;. \qquad \text{(C.12)}$$

Now make a coordinate change to the Hopf fiber coordinates

$$\vartheta=\mu\,, \qquad \xi=\frac{\alpha_1}{2}\,, \qquad \varphi_1=\psi\,, \qquad \varphi_2=\psi+\frac{\alpha_3+\alpha_2}{2}\,, \qquad \varphi_3=\psi+\frac{\alpha_3-\alpha_2}{2}\,, \qquad \text{(C.13)}$$

where α_i are the angles in the three SU(2) left invariant one forms (2.11). The metric becomes

$$ds^2_\beta = \mathcal{H}^{1/2}L^2\left[-r^2\left(1-\frac{r_+^4}{r^4}\right)dt^2 + r^2(dx_1^2+dx_2^2+dy^2) + \left(1-\frac{r_+^4}{r^4}\right)^{-1}\frac{dr^2}{r^2}\right] \tag{C.14}$$
$$+f_0'd\mu^2 + f_1'd\alpha_1^2 + f_2'd\alpha_2^2 + f_3'd\alpha_3^2 + f_4'd\psi^2 + 2f_5'd\alpha_2 d\alpha_3 + 2f_6'd\psi d\alpha_3 + 2f_7'd\psi d\alpha_2.$$

The B-field is

$$B_{(2)} = b_0'd\psi\wedge d\mu + b_1'd\psi\wedge d\alpha_1 + b_2'd\psi\wedge d\alpha_2 + b_3'd\psi\wedge d\alpha_3$$
$$+b_4'd\mu\wedge d\alpha_3 + b_5'd\alpha_1\wedge d\alpha_3 + b_6'd\alpha_2\wedge d\alpha_3. \tag{C.15}$$

The dilaton is

$$\Phi = \frac{1}{2}\ln(\mathcal{G}\mathcal{H}^2)\,, \tag{C.16}$$

where we have defined

$$f_0' = L^2\mathcal{H}^{1/2}\,, \qquad f_1' = L^2\mathcal{H}^{1/2}\,\frac{\sin^2\mu}{4}\,,$$
$$f_2' = L^2\mathcal{H}^{1/2}\mathcal{G}\,\frac{\sin^2\mu}{4}\,, \qquad f_3' = L^2\mathcal{H}^{1/2}\mathcal{G}\,\left(\frac{\sin^2\mu}{4} + |\beta|^2L^4\mu_1^2\mu_2^2\mu_3^2\right)\,,$$
$$f_4' = L^2\mathcal{H}^{1/2}\mathcal{G}\,\left(1+9|\beta|^2L^4\mu_1^2\mu_2^2\mu_3^2\right)\,, \qquad f_5' = L^2\mathcal{H}^{1/2}\mathcal{G}\,\frac{\sin^2\mu\cos\alpha_1}{4}\,,$$
$$f_6' = L^2\mathcal{H}^{1/2}\mathcal{G}\,\left(\frac{\sin^2\mu}{2} + 3|\beta|^2L^4\mu_1^2\mu_2^2\mu_3^2\right)\,, \qquad f_7' = L^2\mathcal{H}^{1/2}\mathcal{G}\,\frac{\sin^2\mu\cos\alpha_1}{2}\,, \tag{C.17}$$

and

$$b_0' = 3\sigma L^4 w_1\,, \qquad b_1' = \frac{3}{2}\sigma L^4 w_2\,, \qquad b_2' = \gamma\mathcal{G}L^4\left(\frac{1}{2}\mu_1^2\mu_2^2 + \frac{1}{2}\mu_1^2\mu_3^2 - \mu_2^2\mu_3^2\right), \tag{C.18}$$
$$b_3' = \gamma\mathcal{G}L^4\,\frac{1}{2}\mu_1^2(\mu_2^2-\mu_3^2)\,, \quad b_4' = -\sigma L^4 w_1\,, \quad b_5' = -\frac{1}{2}\sigma L^4 w_2\,, \quad b_6' = \gamma\mathcal{G}L^4\,\frac{1}{2}\mu_2^2\mu_3^2\,.$$

D Hopf fiber of S^5

The standard metric on the unit radius S^5 is

$$ds^2_{S^5} = \sum_{i=1}^{3}\left(d\mu_i^2 + \mu_i^2 d\varphi_i^2\right) = d\vartheta^2 + \cos^2\vartheta d\varphi_1^2 + \sin^2\vartheta(d\xi^2 + \cos^2\xi d\varphi_2^2 + \sin^2\xi d\varphi_3^2)\,, \tag{D.1}$$

where we have defined

$$\mu_1 = \cos\vartheta\,, \qquad \mu_2 = \sin\vartheta\cos\xi\,, \qquad \mu_3 = \sin\vartheta\sin\xi. \tag{D.2}$$

After the coordinate change (C.13) the metric on S^5 is written as a Hopf fiber over $\mathbb{CP}^2$

$$ds^2_{S^5} = ds^2_{\mathbb{CP}^2} + (d\psi+\mathcal{A})^2 = d\mu^2 + \frac{\sin^2\mu}{4}\left(\sigma_1^2+\sigma_2^2+\cos^2\mu\ \sigma_3^2\right) + \left(d\psi + \frac{\sin^2\mu}{2}\ \sigma_3\right)^2, \tag{D.3}$$

where σ_i are defined in (2.11). Note that the Kähler form on $\mathbb{CP}^2$ is

$$J = \frac{1}{2}\,d\mathcal{A} = \frac{1}{2}\,\sin\mu\cos\mu\ d\mu\wedge\sigma_3 + \frac{1}{4}\,\sigma_1\wedge\sigma_2\,, \qquad \text{with} \qquad \mathcal{A} = \frac{\sin^2\mu}{2}\,\sigma_3\,. \tag{D.4}$$

E T-duality rules

Here we summarize the T-duality transformation rules for type II theories with non-zero RR-flux [63, 64]. We assume that the T-duality is performed along the y-direction.

The NS fields transform under this according to

$$\begin{aligned} \widetilde{g}_{yy} &= \frac{1}{g_{yy}}\,, \qquad \widetilde{g}_{ay} = \frac{B_{ay}}{g_{yy}}\,, \qquad \widetilde{g}_{ab} = g_{ab} - \frac{g_{ay}g_{yb} + B_{ay}B_{yb}}{g_{yy}}\,, \\ \widetilde{B}_{ay} &= \frac{g_{ay}}{g_{yy}}\,, \qquad \widetilde{B}_{ab} = B_{ab} - \frac{g_{ay}B_{yb} + B_{ay}g_{yb}}{g_{yy}}\,, \\ \widetilde{\Phi} &= \Phi - \frac{1}{2}\log g_{yy}\;. \end{aligned} \tag{E.1}$$

The RR fields transform as

$$\begin{aligned} \tilde{C}^{(n)}_{\mu\dots\nu\alpha y} &= C^{(n-1)}_{\mu\dots\nu\alpha} - (n-1)\frac{C^{(n-1)}_{[\mu\dots\nu|y}g_{|\alpha]y}}{g_{yy}}\,, \\ \tilde{C}^{(n)}_{\mu\dots\nu\alpha\beta} &= C^{(n+1)}_{\mu\dots\nu\alpha\beta y} + nC^{(n-1)}_{[\mu\dots\nu\alpha}B_{\beta]y} + n(n-1)\frac{C^{(n-1)}_{[\mu\dots\nu|y}B_{|\alpha|y}g_{|\beta]y}}{g_{yy}}\;. \end{aligned} \tag{E.2}$$

It is also useful to write down the T-duality rules for the RR-fluxes

$$\begin{aligned} \tilde{F}^{(n)}_{\mu_1\dots\mu_{n-1}y} &= F^{(n-1)}_{\mu_1\dots\mu_{n-1}} + (n-1)(-1)^n\frac{g_{y[\mu_1}F^{(n-1)}_{\mu_2\dots\mu_{n-1}]y}}{g_{yy}}\,, \\ \tilde{F}^{(n)}_{\mu_1\dots\mu_n} &= F^{(n+1)}_{\mu_1\dots\mu_n y} - n(-1)^n B_{y[\mu_1}F^{(n-1)}_{\mu_2\dots\mu_n]} - n(n-1)\frac{B_{y[\mu_1}g_{\mu_2|y|}F^{(n-1)}_{\mu_3\dots\mu_n]y}}{g_{yy}}\;. \end{aligned} \tag{E.3}$$

References

[1] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111] [SPIRES].

[2] E. D'Hoker and D.Z. Freedman, *Supersymmetric gauge theories and the AdS/CFT correspondence*, hep-th/0201253 [SPIRES].

[3] D.T. Son, *Toward an AdS/cold atoms correspondence: a geometric realization of the Schroedinger symmetry*, *Phys. Rev.* **D 78** (2008) 046003 [arXiv:0804.3972] [SPIRES].

[4] K. Balasubramanian and J. McGreevy, *Gravity duals for non-relativistic CFTs*, *Phys. Rev. Lett.* **101** (2008) 061601 [arXiv:0804.4053] [SPIRES].

[5] C. Duval, G.W. Gibbons and P. Horvathy, *Celestial Mechanics, Conformal Structures and Gravitational Waves*, *Phys. Rev.* **D 43** (1991) 3907 [hep-th/0512188] [SPIRES].

[6] W.D. Goldberger, *AdS/CFT duality for non-relativistic field theory*, *JHEP* **03** (2009) 069 [arXiv:0806.2867] [SPIRES].

[7] J.L.F. Barbon and C.A. Fuertes, *On the spectrum of nonrelativistic AdS/CFT*, *JHEP* **09** (2008) 030 [arXiv:0806.3244] [SPIRES].

JHEP07(2009)098

[8] C.P. Herzog, M. Rangamani and S.F. Ross, *Heating up Galilean holography*, *JHEP* **11** (2008) 080 [arXiv:0807.1099] [SPIRES].

[9] J. Maldacena, D. Martelli and Y. Tachikawa, *Comments on string theory backgrounds with non- relativistic conformal symmetry*, *JHEP* **10** (2008) 072 [arXiv:0807.1100] [SPIRES].

[10] A. Adams, K. Balasubramanian and J. McGreevy, *Hot Spacetimes for Cold Atoms*, *JHEP* **11** (2008) 059 [arXiv:0807.1111] [SPIRES].

[11] S.S. Pal, *Null Melvin Twist to Sakai-Sugimoto model*, arXiv:0808.3042 [SPIRES].

[12] S. Sekhar Pal, *Towards Gravity solutions of AdS/CMT*, arXiv:0808.3232 [SPIRES].

[13] S. Pal, *More gravity solutions of AdS/CMT*, arXiv:0809.1756 [SPIRES].

[14] C. Duval, M. Hassaine and P.A. Horvathy, *The geometry of Schródinger symmetry in gravity background/non-relativistic CFT*, *Annals Phys.* **324** (2009) 1158 [arXiv:0809.3128] [SPIRES].

[15] S.A. Hartnoll and K. Yoshida, *Families of IIB duals for nonrelativistic CFTs*, *JHEP* **12** (2008) 071 [arXiv:0810.0298] [SPIRES].

[16] L. Mazzucato, Y. Oz and S. Theisen, *Non-relativistic Branes*, *JHEP* **04** (2009) 073 [arXiv:0810.3673] [SPIRES].

[17] M. Taylor, *Non-relativistic holography*, arXiv:0812.0530 [SPIRES].

[18] S.S. Pal, *Anisotropic gravity solutions in AdS/CMT*, arXiv:0901.0599 [SPIRES].

[19] A. Donos and J.P. Gauntlett, *Supersymmetric solutions for non-relativistic holography*, *JHEP* **03** (2009) 138 [arXiv:0901.0818] [SPIRES].

[20] F. Denef and S.A. Hartnoll, *Landscape of superconducting membranes*, *Phys. Rev.* **D 79** (2009) 126008 [arXiv:0901.1160] [SPIRES].

[21] A. Bagchi and R. Gopakumar, *Galilean Conformal Algebras and AdS/CFT*, *JHEP* **07** (2009) 037 [arXiv:0902.1385] [SPIRES].

[22] M. Alishahiha, R. Fareghbal, A.E. Mosaffa and S. Rouhani, *Asymptotic symmetry of geometries with Schrödinger isometry*, arXiv:0902.3916 [SPIRES].

[23] M. Alishahiha, A. Davody and A. Vahedi, *On AdS/CFT of Galilean Conformal Field Theories*, arXiv:0903.3953 [SPIRES].

[24] S. Schäfer-Nameki, M. Yamazaki and K. Yoshida, *Coset Construction for Duals of Non-relativistic CFTs*, arXiv:0903.4245 [SPIRES].

[25] E.O. Colgain and H. Yavartanoo, *NR CFT_3 duals in M-theory*, arXiv:0904.0588 [SPIRES].

[26] S. Kachru, X. Liu and M. Mulligan, *Gravity Duals of Lifshitz-like Fixed Points*, *Phys. Rev.* **D 78** (2008) 106005 [arXiv:0808.1725] [SPIRES].

[27] S.A. Hartnoll, *Lectures on holographic methods for condensed matter physics*, arXiv:0903.3246 [SPIRES].

[28] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, *Renormalization group flows from holography supersymmetry and a c-theorem*, *Adv. Theor. Math. Phys.* **3** (1999) 363 [hep-th/9904017] [SPIRES].

[29] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, *Continuous distributions of D3-branes and gauged supergravity*, *JHEP* **07** (2000) 038 [hep-th/9906194] [SPIRES].

JHEP07(2009)098

[30] R.G. Leigh and M.J. Strassler, *Exactly marginal operators and duality in four-dimensional $N = 1$ supersymmetric gauge theory*, Nucl. Phys. **B 447** (1995) 95 [hep-th/9503121] [SPIRES].

[31] O. Lunin and J.M. Maldacena, *Deforming field theories with* $\mathrm{U}(1) \times \mathrm{U}(1)$ *global symmetry and their gravity duals*, JHEP **05** (2005) 033 [hep-th/0502086] [SPIRES].

[32] e.g. Gimon, A. Hashimoto, V.E. Hubeny, O. Lunin and M. Rangamani, *Black strings in asymptotically plane wave geometries*, JHEP **08** (2003) 035 [hep-th/0306131] [SPIRES].

[33] V.E. Hubeny, M. Rangamani and S.F. Ross, *Causal structures and holography*, JHEP **07** (2005) 037 [hep-th/0504034] [SPIRES].

[34] K. Pilch and N.P. Warner, *$N = 1$ supersymmetric renormalization group flows from IIB supergravity*, Adv. Theor. Math. Phys. **4** (2002) 627 [hep-th/0006066] [SPIRES].

[35] K. Pilch and N.P. Warner, *A new supersymmetric compactification of chiral IIB supergravity*, Phys. Lett. **B 487** (2000) 22 [hep-th/0002192] [SPIRES].

[36] K. Pilch and N.P. Warner, *$N = 2$ supersymmetric RG flows and the IIB dilaton*, Nucl. Phys. **B 594** (2001) 209 [hep-th/0004063] [SPIRES].

[37] A. Brandhuber and K. Sfetsos, *An $N = 2$ gauge theory and its supergravity dual*, Phys. Lett. **B 488** (2000) 373 [hep-th/0004148] [SPIRES].

[38] I.R. Klebanov and E. Witten, *Superconformal field theory on threebranes at a Calabi-Yau singularity*, Nucl. Phys. **B 536** (1998) 199 [hep-th/9807080] [SPIRES].

[39] N. Halmagyi, K. Pilch, C. Romelsberger and N.P. Warner, *Holographic duals of a family of $N = 1$ fixed points*, JHEP **08** (2006) 083 [hep-th/0506206] [SPIRES].

[40] A. Buchel and J.T. Liu, *Thermodynamics of the $N = 2^*$ flow*, JHEP **11** (2003) 031 [hep-th/0305064] [SPIRES].

[41] S. Frolov, *Lax pair for strings in Lunin-Maldacena background*, JHEP **05** (2005) 069 [hep-th/0503201] [SPIRES].

[42] N. Bobev, A. Kundu and K. Pilch, *Supersymmetric IIB Solutions with Schródinger Symmetry*, arXiv:0905.0673 [SPIRES].

[43] D. Marolf and L.A. Pando Zayas, *On the singularity structure and stability of plane waves*, JHEP **01** (2003) 076 [hep-th/0210309] [SPIRES].

[44] D. Brecher, J.P. Gregory and P.M. Saffin, *String theory and the classical stability of plane waves*, Phys. Rev. **D 67** (2003) 045014 [hep-th/0210308] [SPIRES].

[45] M. Cvetič, H. Lü and C.N. Pope, *Geometry of the embedding of supergravity scalar manifolds in $D = 11$ and $D = 10$*, Nucl. Phys. **B 584** (2000) 149 [hep-th/0002099] [SPIRES].

[46] R. Corrado, K. Pilch and N.P. Warner, *An $N = 2$ supersymmetric membrane flow*, Nucl. Phys. **B 629** (2002) 74 [hep-th/0107220] [SPIRES].

[47] L.J. Romans, *New compactifications of Chiral $N = 2$, $d = 10$ supergravity*, Phys. Lett. **B 153** (1985) 392 [SPIRES].

[48] N.J. Evans, C.V. Johnson and M. Petrini, *The enhancon and $N = 2$ gauge theory/gravity RG flows*, JHEP **10** (2000) 022 [hep-th/0008081] [SPIRES].

[49] A. Buchel, A.W. Peet and J. Polchinski, *Gauge dual and noncommutative extension of an $N = 2$ supergravity solution*, Phys. Rev. **D 63** (2001) 044009 [hep-th/0008076] [SPIRES].

JHEP07(2009)098

[50] S.D. Avramis, K. Sfetsos and D. Zoakos, *Complex marginal deformations of D3-brane geometries, their Penrose limits and giant gravitons*, *Nucl. Phys.* **B 787** (2007) 55 [arXiv:0704.2067] [SPIRES].

[51] E. Imeroni, *On deformed gauge theories and their string/M-theory duals*, *JHEP* **10** (2008) 026 [arXiv:0808.1271] [SPIRES].

[52] R. Donagi and E. Witten, *Supersymmetric Yang-Mills Theory And Integrable Systems*, *Nucl. Phys.* **B 460** (1996) 299 [hep-th/9510101] [SPIRES].

[53] J. Polchinski and M.J. Strassler, *The string dual of a confining four-dimensional gauge theory*, hep-th/0003136 [SPIRES].

[54] S. Kachru and E. Silverstein, *4D conformal theories and strings on orbifolds*, *Phys. Rev. Lett.* **80** (1998) 4855 [hep-th/9802183] [SPIRES].

[55] R. Corrado and N. Halmagyi, $N = 1$ *Field Theories and Fluxes in IIB String Theory*, *Phys. Rev.* **D 71** (2005) 046001 [hep-th/0401141] [SPIRES].

[56] N. Halmagyi, K. Pilch, C. Romelsberger and N.P. Warner, *The complex geometry of holographic flows of quiver gauge theories*, *JHEP* **09** (2006) 063 [hep-th/0406147] [SPIRES].

[57] A. Bergman, K. Dasgupta, O.J. Ganor, J.L. Karczmarek and G. Rajesh, *Nonlocal field theories and their gravity duals*, *Phys. Rev.* **D 65** (2002) 066005 [hep-th/0103090] [SPIRES].

[58] A. Bergman and O.J. Ganor, *Dipoles, twists and noncommutative gauge theory*, *JHEP* **10** (2000) 018 [hep-th/0008030] [SPIRES].

[59] K. Dasgupta, O.J. Ganor and G. Rajesh, *Vector deformations of* $N = 4$ *super-Yang-Mills theory, pinned branes and arched strings*, *JHEP* **04** (2001) 034 [hep-th/0010072] [SPIRES].

[60] C.N. Gowdigere and N.P. Warner, *Flowing with eight supersymmetries in M-theory and F-theory*, *JHEP* **12** (2003) 048 [hep-th/0212190] [SPIRES];
C.N. Gowdigere, D. Nemeschansky and N.P. Warner, *Supersymmetric solutions with fluxes from algebraic Killing spinors*, *Adv. Theor. Math. Phys.* **7** (2004) 787 [hep-th/0306097] [SPIRES];
I. Bena and N.P. Warner, *A harmonic family of dielectric flow solutions with maximal supersymmetry*, *JHEP* **12** (2004) 021 [hep-th/0406145] [SPIRES];
C.N. Gowdigere and N.P. Warner, *Holographic Coulomb branch flows with* $N = 1$ *supersymmetry*, *JHEP* **03** (2006) 049 [hep-th/0505019] [SPIRES].

[61] M. Cvetič, H. Lü, D.N. Page and C.N. Pope, *New Einstein-Sasaki spaces in five and higher dimensions*, *Phys. Rev. Lett.* **95** (2005) 071101 [hep-th/0504225] [SPIRES].

[62] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Sasaki-Einstein metrics on* $S^2 \times S^3$, *Adv. Theor. Math. Phys.* **8** (2004) 711 [hep-th/0403002] [SPIRES].

[63] T.H. Buscher, *A Symmetry of the String Background Field Equations*, *Phys. Lett.* **B 194** (1987) 59 [SPIRES]; *Path Integral Derivation of Quantum Duality in Nonlinear* σ*-models*, *Phys. Lett.* **B 201** (1988) 466 [SPIRES].

[64] R.C. Myers, *Dielectric-branes*, *JHEP* **12** (1999) 022 [hep-th/9910053] [SPIRES];
C.V. Johnson, *D-brane primer*, hep-th/0007170 [SPIRES].